Problem 2(a):

\[
\left[ \hat{J}^i, \hat{J}^j \right] \equiv \frac{1}{4} \epsilon^{ik\ell} \epsilon^{jmn} \left[ \hat{J}^{k\ell}, \hat{J}^{jmn} \right] = \langle \langle \text{by eq. (1)} \rangle \rangle \\
= \frac{1}{4} \epsilon^{ik\ell} \epsilon^{jmn} \left( -ig^{km} \hat{J}^{\ell n} + ig^{kn} \hat{J}^{\ell m} + ig^{\ell m} \hat{J}^{kn} - g^{\ell n} \hat{J}^{km} \right) \\
= \epsilon^{ik\ell} \epsilon^{jmn} \times -ig^{km} \hat{J}^{\ell n} \\
= i \hat{J}^{\ell n} \times \left( -g^{km} \epsilon^{ik\ell} \epsilon^{jmn} = +\delta^{km} \epsilon^{ik\ell} \epsilon^{jmn} = \delta^{ij} \delta^{\ell n} - \delta^{in} \delta^{\ell j} \right) \\
= 0 - i \hat{J}^{ji} = +i \hat{J}^{ij} \\
\equiv +i \epsilon^{ijk} \hat{J}^k. \quad (S.1)
\]

\[
\left[ \hat{J}^i, \hat{K}^j \right] \equiv \frac{1}{2} \epsilon^{ik\ell} \left[ \hat{J}^{k\ell}, \hat{J}^{0j} \right] = \langle \langle \text{by eq. (1)} \rangle \rangle \\
= \frac{1}{2} \epsilon^{ik\ell} \left( -ig^{k0} \hat{J}^{\ell j} + ig^{kj} \hat{J}^{\ell 0} + ig^{\ell 0} \hat{J}^{kj} - ig^{\ell j} \hat{J}^{k0} \right) \\
= \frac{1}{2} \epsilon^{ik\ell} \left( 0 - i\delta^{kj} \hat{J}^{\ell 0} + 0 + i\delta^{\ell j} \hat{J}^{k0} \right) \\
\equiv \frac{1}{2} \epsilon^{ik\ell} \left( +i\delta^{kj} \hat{K}^\ell - i\delta^{\ell j} \hat{K}^k \right) \\
= \frac{1}{2} \epsilon^{ij\ell} \hat{K}^\ell - \frac{1}{2} \epsilon^{ikj} \hat{K}^k \\
= \epsilon^{ijk} \hat{K}^k. \quad (S.2)
\]

\[
\left[ \hat{K}^i, \hat{K}^j \right] \equiv \left[ \hat{J}^{0i}, \hat{J}^{0j} \right] = \langle \langle \text{by eq. (1)} \rangle \rangle \\
= -ig^{00} \hat{J}^{ij} + ig^{ij} \hat{J}^{00} + ig^{0j} \hat{J}^{0i} - ig^{ij} \hat{J}^{00} \\
= -i \hat{J}^{ij} + 0 + 0 + 0, \\
\equiv -i \epsilon^{ijk} \hat{J}^k. \quad (S.3)
\]
Problem 2(b):

\[
\begin{align*}
\left[ \hat{V}^i, \hat{J}^j \right] &= \frac{1}{2} \epsilon^{jkl} \left[ \hat{V}^i, \hat{J}^{kl} \right] = \frac{1}{2} \epsilon^{jkl} \left( ig^{ik} \hat{V}^l - ig^{il} \hat{V}^k \right) \\
&= \frac{1}{2} \epsilon^{jkl} \left( -i \delta^{ik} \hat{V}^l + \delta^{il} \hat{V}^k \right) = -\frac{i}{2} \epsilon^{jkl} \hat{V}^l + \frac{i}{2} \epsilon^{jkl} \hat{V}^k \\
&= i \epsilon^{ijk} \hat{V}^k, \\
\left[ \hat{V}^0, \hat{J}^j \right] &= \frac{1}{2} \epsilon^{jkl} \left[ \hat{V}^0, \hat{J}^{kl} \right] = \frac{1}{2} \epsilon^{jkl} \left( ig^{0k} \hat{V}^l - ig^{0l} \hat{V}^k \right) \\
&= 0,
\end{align*}
\]

(S.4)

\[
\begin{align*}
\left[ \hat{V}^i, \hat{K}^j \right] &= \left[ \hat{V}^i, \hat{J}^{0j} \right] = ig^{i0} \hat{V}^j - ig^{ij} \hat{V}^0 \\
&= +i \delta^{ij} \hat{V}^0,
\end{align*}
\]

(S.5)

\[
\begin{align*}
\left[ \hat{V}^0, \hat{K}^j \right] &= \left[ \hat{V}^0, \hat{J}^{0j} \right] = ig^{00} \hat{V}^j - i \delta_{ij} \hat{V}^0 \\
&= +i \hat{V}^j.
\end{align*}
\]

(S.6)

Note that the Hamiltonian of a relativistic theory is a member of a 4-vector multiplet \( \hat{P}^\mu = (\hat{H}, \hat{P}) \) where \( \hat{P} \) is the net momentum operator. Applying the above equations to the \( \hat{P}^\mu \) vectors, we obtain

\[
\begin{align*}
\left[ \hat{P}^i, \hat{J}^j \right] &= i \epsilon^{ijk} \hat{P}^k, \\
\left[ \hat{H}, \hat{J}^j \right] &= 0, \\
\left[ \hat{P}^i, \hat{K}^j \right] &= +i \delta^{ij} \hat{H}, \\
\left[ \hat{H}, \hat{K}^j \right] &= +i \hat{P}^j.
\end{align*}
\]

(S.8)

In particular, the Hamiltonian \( \hat{H} \) commutes with the three angular momenta \( \hat{J}^j \) but it does not commute with the three generators \( \hat{K}^k \) of the Lorentz boosts.

Problem 2(c):

In the ordinary quantum mechanics, it is often said that generators of continuous symmetries must commute with the Hamiltonian operator. However, this is true only for the symmetries that act in a time independent manner — for example, rotating the 3D space by the same angle at all times \( t \). But when the transformation rules of a symmetry depend on time, the Hamiltonian must change to account for this time dependence.
In a Lorentz boost, the transform $x \rightarrow x'$ obviously depends on time, which changes the way the transformed quantum fields such as $\hat{\Phi}^\prime(x, t)$ depend on $t$. Consequently, the Hamiltonian $\hat{H}$ of the theory must change so that the new Heisenberg equations would match the new time dependence. In terms of the generators, this means that the boost generators $\hat{K}^i$ should not commute with the Hamiltonian.

Note that this non-commutativity is not caused by Lorentz boosts affecting the time itself, $t' = L^0_{\mu}x^\mu \neq t$. Even in non-relativistic theories — where the time is absolute — generators of symmetries that affect the other variables in a time-dependent matter do not commute with $\hat{H}$.

Indeed, consider a Galilean transform from one non-relativistic moving frame into another, $x' = x + vt$ but $t' = t$. This is a good symmetry of non-relativistic particles interacting with each other but not subject to any external potential,

$$\hat{H} = \sum_a \frac{1}{2M} \hat{p}^2_a + \frac{1}{2} \sum_{a \neq b} V(\hat{x}_a - \hat{x}_b).$$  \hspace{1cm} (S.9)

A unitary operator $\hat{G}$ realizing a Galilean symmetry acts on coordinate and momentum operators as

$$\hat{G}\hat{x}_a\hat{G}^\dagger = \hat{x}_a + vt, \quad \hat{G}\hat{p}_a\hat{G}^\dagger = \hat{p}_a + Mv,$$  \hspace{1cm} (S.10)

and it also transforms the Hamiltonian into

$$\hat{G}\hat{H}\hat{G}^\dagger = \hat{H} + v \cdot \hat{P}_\text{tot} + \frac{1}{2}M_\text{tot}v^2.$$  \hspace{1cm} (S.11)

In terms of the Galilean boost generators $\hat{K}_G$,

$$\hat{G} = \exp(-iv \cdot \hat{K}_G), \quad [\hat{x}_a, \hat{K}^i_G] = i\delta^i_a \times t, \quad [\hat{p}_a, \hat{K}^i_G] = iM\delta^i_a, \quad [\hat{H}, \hat{K}^i_G] = i\hat{P}^i_\text{tot}.$$  \hspace{1cm} (S.12)

In particular, the $\hat{K}^i_G$ do not commute with the Hamiltonian.
Problem 2(d):
Consider a linear combination $\frac{1}{2}N_{\mu\nu}\hat{J}^{\mu\nu}$ of Lorentz generators with some generic coefficients $N_{\mu\nu} = -N_{\nu\mu}$. The Lorentz symmetries $L(N, \phi) = \exp\left(\frac{i\phi}{2} N_{\mu\nu}\hat{J}^{\mu\nu}\right)$ generated by this combination preserve the momentum $p^\mu$ of the particle state $|p\rangle$ if and only if

$$\exp\left(\frac{i\phi}{2} N_{\mu\nu}\hat{J}^{\mu\nu}\right) \hat{P}^\alpha \exp\left(-\frac{i\phi}{2} N_{\mu\nu}\hat{J}^{\mu\nu}\right) |p\rangle = \hat{P}^\mu |p\rangle. \quad (S.13)$$

For an infinitesimal $\phi$, this condition becomes

$$\left[ \frac{i\phi}{2} N_{\mu\nu}\hat{J}^{\mu\nu}, \hat{P}^\alpha \right] |p\rangle = 0. \quad (S.14)$$

Applying eqs. (3) to the left hand side of this formula gives us

$$\left[ \frac{i\phi}{2} N_{\mu\nu}\hat{J}^{\mu\nu}, \hat{P}^\alpha \right] |p\rangle = \frac{i\phi}{2} N_{\mu\nu} \left( -i g^{\alpha\mu} \hat{P}^\nu + g^{\alpha\nu} \hat{P}^\mu \right) |p\rangle = \phi N^{\alpha\nu} \hat{P}_\nu |p\rangle = \phi N^{\alpha\nu} p_\nu |p\rangle, \quad (S.15)$$

so the condition for the generator $\frac{1}{2}N_{\mu\nu}\hat{J}^{\mu\nu}$ to preserve particle’s momentum $p^\mu$ is simply

$$N^{\alpha\nu} \times p_\nu = 0. \quad (S.16)$$

In 3D terms, $N^{ij} = \epsilon^{ijk} a^k$ and $N^{0k} = -N^{k0} = b^k$ for some 3-vectors $a$ and $b$, the generator in question is

$$\frac{1}{2}N_{\mu\nu}\hat{J}^{\mu\nu} = a \cdot \hat{J} + b \cdot \hat{K}, \quad (S.17)$$

and the condition (S.16) becomes

$$a \times p - b E = 0 \quad \text{and} \quad b \cdot p = 0. \quad (S.18)$$

Actually, the second condition here is redundant, so the general solution is

$$\text{any } a, \quad b = a \times \frac{p}{E}. \quad (S.19)$$

In terms of eq. (S.17), these solutions mean

$$\frac{1}{2}N_{\mu\nu}\hat{J}^{\mu\nu} = a \cdot \hat{J} + \frac{(a \times p) \cdot \hat{K}}{E} = a \cdot \left( \hat{J} + \frac{p}{E} \times \hat{K} \right) \quad \text{for any } a. \quad (S.20)$$

In other words, the Lorentz symmetries preserving the momentum $p^\mu$ have 3 generators,
namely the components of the 3-vector

\[ \mathbf{J} + \frac{p}{E} \times \mathbf{K} \]  

(S.21)

For a particle moving in \( z \) direction at speed \( \beta = \frac{p^z}{E} \), these components are

\[ \hat{J}^x - \beta \hat{K}^y, \quad \hat{J}^y + \beta \hat{K}^x, \quad \text{and} \quad \hat{J}^z. \]  

(S.22)

Naturally, the generators of any symmetry are defined up to linear combinations. Thus, any 3 linearly-independent combination of the operators (S.22) will generate the same little group \( G(p) \) of Lorentz symmetries preserving the momentum \( p^\mu \). In particular, in eq. (4) I have multiplied 2 of the generators (S.22) by the Lorentz slowdown factor \( \gamma \) while leaving \( \hat{J}_z \) as it is. The purpose of this rescaling is to make all 3 generators normalized as angular momenta, with the standard commutation relations with each other. Indeed:

\[
\begin{align*}
\left[ \hat{j}^z, \tilde{J}^x \right] &= \gamma \left[ \hat{j}^z, \hat{j}^x \right] - \beta \gamma \left[ \hat{j}^z, \hat{K}^y \right] \\
&= \gamma \times i \hat{j}^y - \beta \gamma \times (-i \hat{K}^x) \\
&= i \tilde{J}^y, \\
\left[ \hat{j}^z, \tilde{J}^y \right] &= \gamma \left[ \hat{j}^z, \hat{j}^y \right] + \beta \gamma \left[ \hat{j}^z, \hat{K}^x \right] \\
&= \gamma \times (-i \hat{j}^x) - \beta \gamma \times (+i \hat{K}^y) \\
&= -i \tilde{J}^x, \\
\left[ \tilde{J}^x, \tilde{J}^y \right] &= \gamma^2 \left[ \hat{j}^x, \hat{j}^y \right] - \beta \gamma^2 \left[ \hat{K}^y, \hat{j}^y \right] + \beta \gamma^2 \left[ \hat{j}_x, \hat{K}^x \right] - \beta^2 \gamma^2 \left[ \hat{K}^y, \hat{K}^x \right] \\
&= \gamma^2 \times i \hat{j}^z - 0 + 0 - \beta^2 \gamma^2 \times i \hat{j}^z \\
&= i \tilde{j}^z \times (\gamma^2(1 - \beta^2) = 1) = i \tilde{J}^z.
\end{align*}
\]

(S.23)

Problem 2(e):

Eq. (S.22) for the three combinations of Lorentz generators preserving some particle’s momentum does not care if the particle is massive or massless. The only difference is in the overall normalization factors for these generators.
For a massive particle, multiplying the two transverse combinations of \( \hat{J} \) and \( \hat{K} \) by the \( \gamma \) factor makes the three generators normalized as three components of an angular momentum \( \hat{J} \). But for a massless particle \( \gamma = \infty \), so we do not have that option. Instead, we simply leave the generators exactly as in eq. (S.22) for \( \beta = 1 \), hence eq. (5).

Without the \( \gamma \) factors, the commutation relations \([\hat{J}^z, \hat{I}^y] = i\hat{I}^x \) and \([\hat{J}^z, \hat{I}^y] = -i\hat{I}^x \) work exactly as in the first two eqs. (S.23), but in the third equation we get

\[
\left[ \hat{I}^x, \hat{I}^y \right] = i \hat{J}^z \times (1 - \beta^2) \langle \text{without the } \gamma^2 \text{ factor} \rangle = 0. \tag{S.24}
\]

**Problem 2(f):**
The quantum state \( |p, \lambda \rangle \) has definite momentum \( p^\mu \), thus \( \hat{P}_\alpha |p, \lambda \rangle = p_\alpha |p, \lambda \rangle \) and likewise

\[
\epsilon_{\alpha\beta\gamma\delta} \hat{J}^{\beta\gamma} \hat{P}^\delta |p, \lambda \rangle = \epsilon_{\alpha\beta\gamma\delta} p^\delta \hat{J}^{\beta\gamma} |p, \lambda \rangle. \tag{S.25}
\]

For simplicity, let’s assume the particle moves in the \( z \) direction and spell out the operator \( \hat{Q}_\alpha = \epsilon_{\alpha\beta\gamma\delta} p^\delta \hat{J}^{\beta\gamma} \) in components:

\[
\begin{align*}
\hat{Q}_0 &= \epsilon_{0ij} p^3 \times \hat{J}^{ij} = 2p^3 \times \hat{J}^{12} = 2p^z \times J^z, \\
\hat{Q}_1 &= \epsilon_{1023} p^3 \times \hat{J}^{02} + \epsilon_{1203} p^3 \times \hat{J}^{20} + \epsilon_{1jk0} p^0 \times \hat{J}^{jk} \\
&= -2p^3 \times \hat{J}^{02} - 2p^0 \times \hat{J}^{23} = -2p^z \times \hat{K}^y - 2p^0 \times \hat{J}^x, \\
\hat{Q}_2 &= \epsilon_{2013} p^3 \times \hat{J}^{01} + \epsilon_{2103} p^3 \times \hat{J}^{10} + \epsilon_{2jk0} p^0 \times \hat{J}^{jk} \\
&= +2p^3 \times \hat{J}^{01} - 2p^0 \times \hat{J}^{31} = +2p^z \times \hat{K}^x - 2p^0 \times \hat{J}^y, \\
\hat{Q}_3 &= \epsilon_{3ij0} p^0 \times \hat{J}^{ij} = -2p^0 \times \hat{J}^{12} = -2p^0 \hat{J}^z.
\end{align*}
\]

For a massless particle with \( p^z = p^0 = E \), these components simplify to

\[
\begin{align*}
\hat{Q}_0 &= +2E \times \hat{J}^z, \quad \hat{Q}_x = -2E \times \hat{I}^x, \quad \hat{Q}_y = -2E \times \hat{I}^y, \quad \hat{Q}_z = -2E \times \hat{J}^z. \tag{S.27}
\end{align*}
\]

Besides definite momentum, the state \( |p, \lambda \rangle \) has definite helicity \( \lambda \), \( \hat{J}^z |p, \lambda \rangle = \lambda |p, \lambda \rangle \).

Moreover, it is annihilated by other two generators of the little group of its momentum,
\[ \hat{I}^x |p, \lambda\rangle = \hat{I}^y |p, \lambda\rangle = 0. \] Therefore, when we apply the operators (S.27) to this state, we obtain
\[\begin{align*}
\hat{Q}_0 |p, \lambda\rangle &= +2E\lambda |p, \lambda\rangle, \\
\hat{Q}_x |p, \lambda\rangle &= 0, \\
\hat{Q}_y |p, \lambda\rangle &= 0, \\
\hat{Q}_z |p, \lambda\rangle &= -2E\lambda |p, \lambda\rangle,
\end{align*}\] (S.28)
Comparing the right hand sides here to the particle momentum (with a lower index) \( p_\alpha = (+E, 0, 0, -E) \) we see that
\[ \hat{Q}_\alpha |p, \lambda\rangle = 2\lambda p_\alpha |p, \lambda\rangle. \] (S.29)
In other words,
\[ \epsilon_{\alpha\beta\gamma\delta} \hat{J}^\beta \hat{P}^\gamma |p, \lambda\rangle = 2\lambda \hat{P}_\alpha |p, \lambda\rangle. \] (8)

What if a massless particle moves in another direction? In 3-vector notations, eqs. (S.26) become
\[ \begin{align*}
\hat{Q}^0 &= 2E\vec{\beta} \cdot \hat{J}, & \hat{Q} &= 2E(\hat{J} + \vec{\beta} \times \hat{K}).
\end{align*}\] (S.30)
Up the overall factor 2\( E \), the components of \( \hat{Q} \) are the generators (S.22) of the little group of the momentum \( p^\mu \). The definite-helicity state \( |p, \lambda\rangle \) is annihilated by the two generators \( \perp \cdot p \) and is an eigenstate of the third generator \( \parallel \cdot p \). Indeed, for a massless particle \( \vec{\beta} \cdot \hat{J} \) is the helicity operator, hence
\[ \begin{align*}
\hat{Q}^0 |p, \lambda\rangle &= 2E\lambda |p, \lambda\rangle, & \vec{\beta} \cdot \hat{Q} |p, \lambda\rangle &= 2E\vec{\beta} \cdot \hat{J} |p, \lambda\rangle = 2E\lambda |p, \lambda\rangle,
\end{align*}\] (S.31)
while
\[ \vec{\beta} \times \hat{Q} |p, \lambda\rangle = 0. \] (S.32)
Consequently,
\[ \hat{Q} |p, \lambda\rangle = \vec{\beta} \left( \vec{\beta} \cdot \hat{Q} |p, \lambda\rangle \right) - \vec{\beta} \times \vec{\beta} \times \hat{Q} |p, \lambda\rangle = \vec{\beta} (2E\lambda |p, \lambda\rangle) = 2p\lambda |p, \lambda\rangle, \] (S.33)
and combining this formula with $\hat{Q}^0 |p, \lambda\rangle = 2E\lambda |p, \lambda\rangle$ we obtain

$$\hat{Q}^\mu |p, \lambda\rangle = 2\lambda p^\mu |p, \lambda\rangle.$$  \hfill (S.34)

This proves eq. (8) for a massless particle moving in any direction.

**Problem 2(g):**

Consider a continuous Lorentz transform $x^\mu \rightarrow x'^\mu = L^\mu_{\nu}x^\nu$ acting on the $|p, \lambda\rangle$ state of a massless particle. The operators on both sides of both sides of eq. (8) transform as Lorentz vectors,

$$\hat{D}(L)\hat{P}_\alpha\hat{D}^\dagger(L) = L^\mu_{\alpha}\hat{P}_\mu, \quad \hat{D}(L)\left(\epsilon_{\alpha\beta\gamma\delta}\hat{j}^{\beta\gamma}\hat{P}^\delta\right)\hat{D}^\dagger(L) = L^\mu_{\alpha}\left(\epsilon_{\mu\beta\gamma\delta}\hat{j}^{\beta\gamma}\hat{P}^\delta\right).$$ \hfill (S.35)

Consequently, the transformed state

$$\hat{D}|p, \lambda\rangle = |Lp, ??\rangle$$ \hfill (S.36)

satisfies the same eq. (8) as the original state $|p, \lambda\rangle$. Indeed,

$$L^\mu_{\alpha}\left(\epsilon_{\mu\beta\gamma\delta}\hat{j}^{\beta\gamma}\hat{P}^\delta\right)|Lp, ??\rangle = \hat{D}(L)\left(\epsilon_{\alpha\beta\gamma\delta}\hat{j}^{\beta\gamma}\hat{P}^\delta\right)\hat{D}^\dagger(L) \times \hat{D}|p, \lambda\rangle$$

$$= \hat{D}(L) \times \left(\epsilon_{\alpha\beta\gamma\delta}\hat{j}^{\beta\gamma}\hat{P}^\delta\right)|p, \lambda\rangle$$ \hfill (S.37)

$$\ll \text{by eq. (8)} \rr$$

$$= \hat{D}(L) \times 2\lambda \hat{P}_\alpha|p, \lambda\rangle$$

$$= 2\lambda \times \hat{D}(L)\hat{P}_\alpha\hat{D}^\dagger(L) \times \hat{D}|p, \lambda\rangle$$

$$= 2\lambda \times L^\mu_{\alpha}\hat{P}_\mu|Lp, ??\rangle,$$

and hence

$$\left(\epsilon_{\mu\beta\gamma\delta}\hat{j}^{\beta\gamma}\hat{P}^\delta\right)|Lp, ??\rangle = 2\lambda \hat{P}_\mu|Lp, ??\rangle.$$ \hfill (S.38)

Note that this equation for the transformed state $|Lp, ??\rangle$ has exactly the same helicity eigenvalue $\lambda$ as the original eq. (8). In other words, the transformed state has the same
helicity as the original state, and since the momentum and the helicity completely determine the quantum state of a particle up to an overall phase, it follows that

\[ \hat{D}(L) \left| p, \lambda \right\rangle = \left| Lp, \text{same } \lambda \right\rangle \times e^{i \text{phase}}. \] (9)

For massless particles, the continuous Lorentz transforms preserve helicity!

**Problem 3(a):**

In light of eqs. (2),

\[ \left[ \hat{j}_i^\pm, \hat{j}_j^\pm \right] = \frac{1}{4} \left[ \hat{j}_i^i, \hat{j}_j^j \right] \pm \frac{i}{4} \left[ \hat{K}_i^i, \hat{K}_j^j \right] - \frac{1}{4} \left[ \hat{K}_i^i, \hat{K}_j^j \right] \]

\[ = \frac{1}{4} i \epsilon^{ijk} \hat{j}_k^j \pm \frac{i}{4} i \epsilon^{ijk} \hat{K}_k^k \pm \frac{i}{4} i \epsilon^{ijk} \hat{K}_j^k \]

\[ = \epsilon^{ijk} \left( \frac{1}{2} \hat{j}_k^j \mp \frac{1}{2} \hat{K}_k^k \right) = i \epsilon^{ijk} \hat{j}_k^j, \] (S.39)

\[ \left[ \hat{j}_i^i, \hat{j}_j^j \right] = \frac{1}{4} \left[ \hat{j}_i^i, \hat{j}_j^j \right] \pm \frac{i}{4} \left[ \hat{K}_i^i, \hat{K}_j^j \right] \pm \frac{i}{4} \left[ \hat{j}_i^i, \hat{K}_j^j \right] + \frac{1}{4} \left[ \hat{K}_i^i, \hat{K}_j^j \right] \]

\[ = \frac{1}{4} i \epsilon^{ijk} \hat{j}_k^j \pm \frac{i}{4} i \epsilon^{ijk} \hat{K}_k^k \pm \frac{i}{4} i \epsilon^{ijk} \hat{K}_j^k + \frac{1}{4} (-i) \epsilon^{ijk} \hat{j}_k^j \]

\[ = 0. \]

**Problem 3(b):**

All three Pauli matrices \( \sigma_i \) are hermitian; specifically, the \( \sigma_1 \) and the \( \sigma_3 \) are real and symmetric while the \( \sigma_2 \) is imaginary and antisymmetric. Consequently, all three matrices are related to their complex conjugates as

\[ \sigma_2 \times \sigma_i^\dagger \times \sigma_2 = -\sigma_i. \] (S.40)

Indeed, by inspection

\[ \sigma_2 \sigma_1^\dagger \sigma_2 = +\sigma_2 \sigma_1 \sigma_2 = -\sigma_2 \sigma_2 \times \sigma_1 = -\sigma_1, \] (S.41)

\[ \sigma_2 \sigma_2^\dagger \sigma_2 = -\sigma_2 \sigma_2 \sigma_2 = -\sigma_2 \sigma_2 \times \sigma_2 = -\sigma_2, \] (S.42)

\[ \sigma_2 \sigma_3^\dagger \sigma_2 = +\sigma_2 \sigma_3 \sigma_2 = -\sigma_2 \sigma_2 \times \sigma_3 = -\sigma_3. \] (S.43)
Now, let $a$ and $b$ be two real 3-vectors parametrizing a Lorentz symmetry $\exp(-ia \cdot \hat{J} - ib \cdot \hat{K})$. Let $A$ be the exponent in eq. (11),

$$A = \frac{1}{2}(-ia + b) \cdot \sigma \implies M(L) = \exp(A). \quad (S.44)$$

Then in light of eq. (S.40)

$$\sigma_2 \times A^* \times \sigma_2 = \frac{1}{2}(-ia + b)^* \cdot \sigma_2 \sigma^* \sigma_2 = \frac{1}{2}(+ia + b) \cdot (-\sigma) = \frac{1}{2}(-ia - b) \cdot \sigma, \quad (S.45)$$

where the RHS is the exponent in eq. (12), Thus,

$$M(L) = \exp(\sigma_2 \times A^* \times \sigma_2) = \sigma_2 \times (\exp(A))^* \times \sigma_2 = \sigma_2 \times M^* \times \sigma_2 \quad (S.46)$$

where the second equality follows from expanding the exponential into the power series. Indeed, since $\sigma_2 \sigma_2 = 1$, we have

$$(\sigma_2 A^* \sigma_2)^2 = \sigma_2 A^* \sigma_2 \times A^* \sigma_2 = A^* A^* \sigma_2 \sigma_2 \implies \sigma_2 \times (A^* \sigma_2)^2 \times \sigma_2 = A^* (A^2)^* \times \sigma_2 \quad (S.47)$$

and likewise for any integer power $n$,

$$(\sigma_2 A^* \sigma_2)^n = \sigma_2 \times (A^n)^* \times \sigma_2, \quad (S.48)$$

hence

$$\exp(\sigma_2 A^* \sigma_2) = \sum_n \frac{1}{n!} (\sigma_2 A^* \sigma_2)^n = \sigma_2 \times \left( \sum_n \frac{1}{n!} (A^n)^2 \right) \times \sigma_2 = \sigma_2 \times (\exp(A))^* \times \sigma_2. \quad (S.49)$$

This completes the proof of $M = \sigma_2 \times M^* \times \sigma_2$. 

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The other equality in eq. (13) is even easier. By hermiticity of the Pauli matrices,

$$-A^\dagger = \frac{1}{2}(-ia - b) \cdot \sigma,$$

which is the exponent in eq. (12). Thus,

$$\overline{M} = \exp(-A^\dagger) \implies \overline{M}^{-1} = \exp(+A^\dagger) = (\exp(A))^\dagger = M^\dagger,$$

and therefore

$$\overline{M} = (M^\dagger)^{-1}.$$

Q.E.D.

Problem 3(c):
Let’s start with reality. The matrix $V = V_\mu \sigma^\mu$ is hermitian if and only if the 4-vector $V^\mu$ is real. For any matrix $M \in SL(2, \mathbb{C})$, the transform

$$V \rightarrow V' = MVM^\dagger$$

preserves hermiticity: if $V$ is hermitian, then so is $V'$; indeed

$$(V')^\dagger = (MVM^\dagger)^\dagger = (M^\dagger)^\dagger V^\dagger M^\dagger = MVM^\dagger = V'. $$

In terms of the 4-vectors, this means that if $V^\mu$ is real than $V'^\mu = L^\mu_\nu V^\nu$ is also real. In other words, the $4 \times 4$ matrix $L^\mu_\nu(M)$ is real.

Next, let’s prove that $L^\mu_\nu(M) \in O(3,1)$ — it preserves the Lorentz metric $g^{\alpha\beta}$, or equivalently, for any $V^\mu$, $g^{\alpha\beta} V'^\alpha V'^\beta = g^{\alpha\beta} V^\alpha V^\beta$. In $2 \times 2$ matrix terms, the Lorenz square of
a 4-vector becomes the determinant:

$$g^{\alpha\beta} V_{\alpha} V_{\beta} = \det(V = V_\mu \sigma^\mu). \quad (S.55)$$

Indeed, from the explicit form of the 4 matrices

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \quad (S.56)$$

we have

$$V = V_\mu \sigma^\mu = \begin{pmatrix} V_0 + V_3 & V_1 - iV_2 \\ V_1 + iV_2 & V_0 - V_3 \end{pmatrix}$$

and hence

$$\det(V) = (V_0 + V_3)(V_0 - V_3) - (V_1 - iV_2)(V_1 + iV_2) = V_0^2 - V_3^2 - V_1^2 - V_2^2 = g^{\alpha\beta} V_{\alpha} V_{\beta}. \quad (S.57)$$

The determinant of a matrix product is the product of the individual matrices’ determinants. Hence, for the transform (S.53),

$$\det(V') = \det(M) \times \det(V) \times \det(M^\dagger) = \det(V) \times |\det(M)|^2. \quad (S.58)$$

The $M$ matrices of interest to us belong to the $SL(2, \mathbb{C})$ group — they are complex matrices with units determinants. There are no other restrictions, but $\det(M) = 1$ is enough to assure $\det(V') = \det(V)$, cf. eq. (S.58). Thanks to the relation (S.55), this means

$$g^{\alpha\beta} V'_{\alpha} V'_{\beta} = \det(V') = \det(V) = g^{\alpha\beta} V_{\alpha} V_{\beta} \quad (S.59)$$

— which proves that the matrix $L^\mu_\nu(M)$ is indeed Lorentzian.

To prove that the Lorentz transform $L^\mu_\nu(M)$ is orthochronous, we need to show that for any $V_\mu$ in the forward light cone — $V^2 > 0$ and $V_0 > 0$ — the $V'_\mu$ is also in the forward light cone. In matrix terms, $V^2 > 0$ and $V_0 > 0$ mean $\det(V) > 0$ and $\text{tr}(V) > 0$; together,
these two conditions means that the $2 \times 2$ hermitian matrix $V$ is positive-definite. The transform (S.53) preserves positive definiteness: if for any complex 2-vector $\xi \neq 0$ we have $\xi^\dagger V \xi > 0$, then
\[
\xi^\dagger V' \xi = \xi^\dagger MVM^\dagger \xi = (M^\dagger \xi)^\dagger V (M^\dagger \xi) > 0.
\] (S.60)
(Note that $M^\dagger \xi \neq 0$ for any $\xi \neq 0$ because det$(M) \neq 0$.) Thus, for any $M \in SL(2, \mathbb{C})$ the Lorentz transform $V^\mu \rightarrow V'^\mu$ preserves the forward light cone — in other words, the $L^\mu_\nu(M)$ is orthochronous, $L^\mu_\nu(M) \in O^+(3, 1)$.

**Problem 3(c*)**:
The simplest proof the $L^\mu_\nu(M)$ is proper as well as orthochronous involves the group law (part (d) of this problem) and the explicit examples of a pure rotation and a pure boost (in parts (e) and (f)), both of which are manifestly proper.

For any $SL(2, \mathbb{C})$ matrix $M$ we may decompose $M = HU$ where $H = \sqrt{MM^\dagger}$ is hermitian and $U = H^{-1}M$ is unitary. (Proof: $UU^\dagger = H^{-1}MM^\dagger H^{-1} = H^{-1}H^2H^{-1} = 1$.) Furthermore, both $H$ and $U$ are unimodular (det$(H) = \det(U) = 1$), or in other words $H, U \in SL(2, \mathbb{C})$, which allows us to define two separate Lorentz transforms $L(H)$ and $L(U)$. According to the group law, together these two transform accomplish the $L(M)$ transform,
\[
L(M) = L(H) \times L(U).
\] (S.61)

Now, $H$ is hermitian, unimodular, and positive definite, hence it has a well-defined logarithm which is hermitian and traceless, $\text{tr}(\log H) = \log(\det(H)) = 0$. For the $2 \times 2$ matrices, this means $\log H = -\frac{1}{2} r \mathbf{\sigma}$ for some real 3-vector $r$, or equivalently $H = \exp\left(-\frac{1}{2} r \mathbf{n} \mathbf{\sigma}\right)$. As we shall see in part (f) below, this means that $L(H)$ is a pure Lorentz boost of rapidity $r$ in the direction $\mathbf{n}$. This boost manifestly does not invert space or time, thus $L(U)$ is proper.

Likewise, $U$ is unitary and unimodular, thus $U \in SU(2)$ and defines a pure rotation of space. Indeed, any $U \in SU(2)$ can be written as $U = \exp\left(-\frac{i}{2} \theta \mathbf{n}' \mathbf{\sigma}\right)$ for some angle $\theta$ and some axis $\mathbf{n}'$, and we shall see in part (e) that for such $U$, $L(U)$ is indeed a rotation of the 3D space by angle $\theta$ around axis $\mathbf{n}'$. Again, this rotation is proper — it does not invert space or time. Thus, $L(H)$ and $L(U)$ are both proper Lorentz transforms, hence their product $L(M)$ must also be proper. (Proof: $\det(L(M)) = \det(L(H)) \times \det(L(U)) = +1.$) \hspace{1cm} Q.E.D.
And by the way, since any proper, orthochronous Lorentz transform \( L \in SO^+(1, 3) \) can be realized as \( L(M) \) for some \( M \in SL(2, \mathbb{C}) \), it follows that any such transform is a product of a pure space rotation \( L(H) \) followed by a pure Lorentz boost \( L(U) \).

Problem 3(d):
Let’s plug \( L_{\mu}^\nu(M) V_\nu = V_\mu' \) into eqs. (20):

\[
\sigma^\mu L_{\mu}^\nu(M) V_\nu = \sigma^\mu V_\mu' = M(\sigma^\nu V_\nu) M^\dagger = M \sigma^\nu M^\dagger \times V_\nu. \tag{S.62}
\]

Since this equation holds true for any 4-vector \( V_\nu \), we must have

\[
\sigma^\mu L_{\mu}^\nu(M) = M \sigma^\nu M^\dagger. \tag{S.63}
\]

This formula defines the Lorentz transform \( L(M) \) in a \( V_\mu \)-independent way, which is more convenient for verifying the group law. Indeed, for any two \( SL(2, \mathbb{C}) \) matrices \( M_1 \) and \( M_2 \), it gives us

\[
\sigma^\mu L_{\mu}^\nu(M_2 M_1) = (M_2 M_1) \sigma^\nu(M_2 M_1) M_2^\dagger = M_2 M_1 \sigma^\nu(M_2 M_1) M_2^\dagger M_1^\dagger M_2^\dagger = M_2 \left( M_1 \sigma^\nu M_1^\dagger \right) M_2^\dagger = M_2 \sigma^\rho M_2^\dagger \times L_\rho^\nu(M_1) = \sigma^\mu L_\mu^\rho(M_2) \times L_\rho^\nu(M_1) = \sigma^\mu \left( L_\mu^\rho(M_2) L_\rho^\nu(M_1) \right) \tag{S.64}
\]

and hence

\[
L_{\mu}^\nu(M_2 M_1) = L_{\mu}^\rho(M_2) L_\rho^\nu(M_1). \tag{S.65}
\]

In short, \( L(M_2 M_1) = L(M_2) L(M_1) \). \( \Box \)
Problem 3(e):
Let \( M = \exp\left(-\frac{i}{2}\theta \mathbf{n}\sigma\right) = \cos \frac{\theta}{2} - i \sin \frac{\theta}{2} \mathbf{n}\sigma \) and hence \( M^\dagger = M^{-1} = \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \mathbf{n}\sigma \). Since \( \sigma^0 = 1 \) while the Pauli matrices \( \sigma^1, 2, 3 \) are traceless, for any unitary \( M \)

\[
M\sigma^0 M^\dagger = MM^\dagger = \sigma^0 \quad (S.66)
\]

while

\[
M\sigma M^\dagger = M\sigma M^{-1} \quad \implies \quad \text{tr}(M\sigma^i M^\dagger) = \text{tr}(M\sigma^i M^{-1}) = \text{tr}(\sigma^i) = 0 \quad (S.67)
\]

In terms of the Lorentz transform (20), this means that \( V_0' = V_0 \) while \( V' \) depends only on the space components of the \( V \). In other words, this Lorentz transform is a rotation of space that does not affect the time.

Specifically,

\[
\mathbf{\sigma} \cdot \mathbf{V}' = M(\mathbf{V}\sigma)M^\dagger = \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \mathbf{n}\sigma\right) (\mathbf{V}\sigma) \left(\cos \frac{\theta}{2} - i \sin \frac{\theta}{2} \mathbf{n}\sigma\right)
\]

\[
= \cos^2 \frac{\theta}{2} (\mathbf{V}\sigma) - i \sin \frac{\theta}{2} \cos \frac{\theta}{2} \left(\mathbf{n}\sigma, \mathbf{V}\sigma\right) + \sin^2 \frac{\theta}{2} \left(\mathbf{n}\sigma(\mathbf{V}\sigma)\mathbf{n}\sigma\right) = 2i(\mathbf{n} \times \mathbf{V}) \cdot \mathbf{\sigma} \quad (S.68)
\]

\[
= \cos \theta (\mathbf{V}\sigma) + \sin \theta ((\mathbf{n} \times \mathbf{V})\sigma) + (1 - \cos \theta)(\mathbf{nV})(\mathbf{n}\sigma),
\]

hence

\[
\mathbf{V}' = \cos \theta(\mathbf{V} - \mathbf{n}(\mathbf{nV})) + \sin \theta \mathbf{n} \times \mathbf{V} + \mathbf{n}(\mathbf{nV}). \quad (S.69)
\]

This is indeed a rotation through angle \( \theta \) around axis \( \mathbf{n} \).
Problem 3(f):
Now consider the Lorentz transforms $L(M)$ for a hermitian matrix $M = M^\dagger$. Specifically, plugging $M = \exp(-\frac{r}{2} \, n\sigma) = \cosh \frac{r}{2} - \sinh \frac{r}{2} \, n\sigma$ into eq. (20), we obtain

$$
\sigma^\mu V'_\mu = V'_0 - \sigma \cdot V' = M(V_0 - \sigma \cdot V)M^\dagger
$$
$$
= \left(\cosh \frac{r}{2} - \sinh \frac{r}{2} \, n\sigma\right) (V_0 - V \sigma) \left(\cosh \frac{r}{2} - \sinh \frac{r}{2} \, n\sigma\right)
$$
$$
= \cosh^2 \frac{r}{2} (V_0 - V \sigma)
- \sinh \frac{r}{2} \cosh \frac{r}{2} \left\{n\sigma, (V_0 - V \sigma)\right\} = 2V_0(n\sigma) - 2(nV)
+ \sinh^2 \frac{r}{2} \left((n\sigma)(V_0 - V \sigma)(n\sigma) = V_0 - 2(nV)(n\sigma) + (V \sigma)\right)
$$
$$
= (\cosh r V_0 + \sinh r \, nV) - (\sigma n)(\sinh r V_0 + \cosh r \, nV)
- \sigma \cdot (V - n(nV)).
$$
(S.70)

In other words,

$$
V'_0 = (\cosh r) V_0 + (\sinh r) \, nV, \quad V' = n((\sinh r) V_0 + (\cosh r) \, nV) + (V - n(nV)),
$$
(S.71)

which is precisely the Lorentz boost of rapidity $r$ in the direction $n$. (The rapidity $r$ is related to the usual parameters of a Lorentz boost according to $\beta = \tanh r$, $\gamma = \cosh r$, $\gamma\beta = \sinh r$. For several boosts in the same directions, the rapidities add up, $r_{\text{tot}} = r_1 + r_2 + \cdots$.) Q.E.D.

Problem 3(g):
For any Lie algebra equivalent to an angular momentum or its analytic continuation, the product of two doublets comprises a triplet and a singlet, $2 \otimes 2 = 3 \oplus 1$, or in $(j)$ notations, $(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2}) = (1) \oplus (0)$. Furthermore, the triplet $3 = (1)$ is symmetric with respect to permutations of the two doublets while the singlet $1 = (0)$ is antisymmetric.

For two separate and independent types of angular momenta $J_+$ and $J_-$ we combine the $j_+$ quantum numbers independently from the $j_-$ and the $j_-$ quantum numbers independently from the $j_+$. For two bi-spinors, this gives us

$$
(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2}) = (1, 1) \oplus (1, 0) \oplus (0, 1) \oplus (0, 0).
$$
(S.72)

Furthermore, the symmetric part of this product should be either symmetric with respect to
both the $j_+$ and the $j_-$ indices or antisymmetric with respect to both indices, thus

$$
[(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2})]_{\text{sym}} = (1, 1) \oplus (0, 0).
$$

Likewise, the antisymmetric part is either symmetric with respect to the $j_+$ but antisymmetric with respect to the $j_-$ or the other way around, thus

$$
[(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2})]_{\text{antisym}} = (1, 0) \oplus (0, 1).
$$

From the $SO^+(3, 1)$ point of view, the bi-spinor $(\frac{1}{2}, \frac{1}{2})$ is the Lorentz vector. A general 2-index Lorentz tensor transforms like a product of two such vectors, so from the $SL(2, \mathbb{C})$ point of view it’s a product of two bi-spinors, which decomposes to irreducible multiplets according to eq. (S.72).

The Lorentz symmetry respects splitting of a general 2-index tensor into a symmetric tensor $T^{\mu \nu} = +T^{\nu \mu}$ and an asymmetric tensor $F^{\mu \nu} = -F^{\nu \mu}$. The symmetric tensor corresponds to a symmetrized square of a bi-spinor, which decomposes into irreducible multiplets according to eq. (S.73). The singlet $(0, 0)$ component is the Lorentz-invariant trace $T^{\mu \mu}$ while the $(1, 1)$ irreducible multiplet is the traceless part of the symmetric tensor.

Likewise, the antisymmetric Lorentz tensor $F^{\mu \nu} = -F^{\nu \mu}$ decomposes according to eq. (S.74). Here, the irreducible components $(1, 0)$ and $(0, 1)$ are complex but conjugate to each other; individually, they describe antisymmetric tensors subject to complex duality conditions $\frac{i}{2} \epsilon^{\kappa \lambda \mu \nu} F_{\mu \nu} = \pm i F^{\kappa \lambda}$, or in 3D terms, $E = \pm i B$. 