Problem 1(a):

\[
\gamma^\alpha \gamma_\alpha = \frac{1}{2} \{\gamma^\alpha, \gamma^\beta\} g_{\alpha\beta} = g^{\alpha\beta} g_{\alpha\beta} = 4; \quad (S.1)
\]

\[
\gamma^\alpha \gamma^\nu \gamma_\alpha = (\gamma^\alpha \gamma^\nu = 2g^{\mu\alpha} - \gamma^\nu \gamma^\alpha) \gamma_\alpha \\
= 2\gamma^\nu - \gamma^\nu (\gamma^\alpha \gamma_\alpha = 4) = -2\gamma^\nu; \quad (S.2)
\]

\[
\gamma^\alpha \gamma^\mu \gamma^\nu \gamma_\alpha = (\gamma^\alpha \gamma^\mu = 2g^{\mu\alpha} - \gamma^\mu \gamma^\alpha) \gamma^\nu \gamma_\alpha \\
= 2\gamma^\nu \gamma^\mu - \gamma^\mu (\gamma^\alpha \gamma^\nu \gamma_\alpha = -2\gamma^\nu) \\
= 2\gamma^\nu \gamma^\mu + 2\gamma^\mu \gamma^\nu = 4g^{\mu\nu}; \quad (S.3)
\]

\[
\gamma^\alpha \gamma^\lambda \gamma^\mu \gamma^\nu \gamma_\alpha = (\gamma^\alpha \gamma^\lambda = 2g^{\mu\alpha} - \gamma^\lambda \gamma^\alpha) \gamma^\mu \gamma^\nu \gamma_\alpha \\
= 2\gamma^\mu \gamma^\nu \gamma^\lambda - \gamma^\lambda (\gamma^\alpha \gamma^\mu \gamma^\nu \gamma_\alpha = 4g^{\mu\nu}) \\
= (2\gamma^\mu \gamma^\nu - 4g^{\mu\nu} = -2\gamma^\nu \gamma^\mu) \gamma^\lambda \\
= -2\gamma^\nu \gamma^\mu \gamma^\lambda. \quad (S.4)
\]

Problem 1(b):

First, a Lemma: for any Lorentz vector \(a^\mu\), the \(\not{a} \equiv \gamma^\mu a_\mu\) matrix squares to

\[
\not{a} \not{a} = \gamma^\mu a_\mu \gamma^\nu a_\nu = a_\mu a_\nu \times \left(\gamma^\mu \gamma^\nu = g^{\mu\nu} - 2iS^{\mu\nu}\right) = a^2 - i[a_\mu, a_\nu] \times S^{\mu\nu} \quad (S.5)
\]

(overlap equality comes from \(S^{\mu\nu} = -S^{\nu\mu}\).) For vectors \(a^\mu\) whose components commute with each other, this formula simplifies to \(\not{a}^2 = a^2\), in particular for the ordinary derivatives \(\partial^\mu, \partial^2 = \partial^2\). However, the covariant derivatives \(D_\mu\) do not commute with each other. Instead,

\[
[D_\mu, D_\nu] = iq F_{\mu\nu}(x) \text{ where } q \text{ is the electric charge of the field on which } D_\mu \text{ act; for the electron field } \Psi(x), \text{ } q = -e \text{ and hence } [D_\mu, D_\nu] \Psi(x) = -ie F_{\mu\nu}(x) \Psi(x). \text{ Hence, according to the lemma (S.5),}
\]

\[
\not{D}^2 \Psi = D^2 \Psi - e F_{\mu\nu} S^{\mu\nu} \Psi. \quad (S.6)
\]

Now, suppose the electron field \(\Psi(x)\) satisfies the covariant Dirac equation \((i \not{D} - m)\Psi = 0\).
Then for any differential operator $\mathcal{D}$, $\mathcal{D} \times (i\not\!\partial - m)\Psi = 0$, and in particular

$$(-i\not\!\partial - m) \times (i\not\!\partial - m)\Psi = 0.$$  \hspace{1cm} (S.7)

The LHS of this formula amounts to

$$(-i\not\!\partial - m) \times (i\not\!\partial - m)\Psi = (\not\!\partial^2 + m^2)\Psi = (D^2 - eF_{\mu\nu}S^{\mu\nu} + m^2)\Psi,$$  \hspace{1cm} (S.8)

which immediately leads to eq. (2).

Problem 1(c):
The anti-commutation relations (1) imply $\gamma^\mu\gamma^\nu = \pm \gamma^\nu\gamma^\mu$ where the sign is ‘+’ for $\mu = \nu$ and ‘−’ otherwise. Hence for any product $\Gamma$ of the $\gamma$ matrices, $\gamma^\mu\Gamma = (-1)^n\Gamma\gamma^\mu$, where $n$ is the number of $\gamma^{\nu\neq\mu}$ factors of $\Gamma$. For $\Gamma = \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$, $n = 3$ for any $\mu = 0, 1, 2, 3$, hence $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$.

As to the spin matrices, $\gamma^5\gamma^\mu\gamma^\nu = -\gamma^\mu\gamma^5\gamma^\nu = +\gamma^\mu\gamma^\nu\gamma^5$ and therefore $\gamma^5 S^{\mu\nu} = +S^{\mu\nu}\gamma^5$.

Problem 1(d):
First, the hermiticity:

$$\left(\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3\right)^\dagger = -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger = +i\gamma^3\gamma^2\gamma^1\gamma^0$$
$$= +i((\gamma^3\gamma^2)\gamma^1)\gamma^0 = (-1)^3i\gamma^0((\gamma^3\gamma^2)\gamma^1)$$
$$= (-1)^{3+2}i\gamma^0(\gamma^1(\gamma^3\gamma^2)) = (-1)^{3+2+1}i\gamma^0(\gamma^1(\gamma^2\gamma^3))$$
$$= +i\gamma^0\gamma^1\gamma^2\gamma^3 \equiv +\gamma^5.$$ \hspace{1cm} (S.9)

Second, the square:

$$\left(\gamma^5\right)^2 = \gamma^5(\gamma^5)^\dagger = (i\gamma^0\gamma^1\gamma^2\gamma^3)(i\gamma^3\gamma^2\gamma^1\gamma^0) = -\gamma^0\gamma^1\gamma^2(\gamma^3\gamma^3)\gamma^2\gamma^1\gamma^0$$
$$= +\gamma^0\gamma^1(\gamma^2\gamma^2)\gamma^1\gamma^0 = -\gamma^0(\gamma^1\gamma^1)\gamma^0 = +\gamma^0\gamma^0 = +1.$$ \hspace{1cm} (S.10)
Problem 1(e): Since the four Dirac matrices $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ all anticommute with each other,

$$\epsilon_{\kappa\lambda\mu\nu} \gamma^\kappa \gamma^\lambda \gamma^\mu \gamma^\nu = \gamma^{[0} \gamma^{1} \gamma^{2} \gamma^{3]} = 24 \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -24 i \gamma^5. \quad (S.11)$$

To prove the other identity, we note that a totally antisymmetric product $\gamma^{[\kappa \gamma^\lambda \gamma^\mu \gamma^\nu]}$ vanishes unless the Lorentz indices $\kappa, \lambda, \mu, \nu$ are all distinct — which makes them $0, 1, 2, 3$ in some order. For such indices, the anticommutativity of Dirac matrices implies

$$\gamma^{[\kappa \gamma^\lambda \gamma^\mu \gamma^\nu]} = -24 \epsilon_{\kappa\lambda\mu\nu} \times \gamma^0 \gamma^1 \gamma^2 \gamma^3 \quad (note \ that \ \epsilon^{0123} = -1),$$

and hence

$$\gamma^{[\kappa \gamma^\lambda \gamma^\mu \gamma^\nu]} = -24 \epsilon_{\kappa\lambda\mu\nu} \times \gamma^0 \gamma^1 \gamma^2 \gamma^3 = +24 i \epsilon_{\kappa\lambda\mu\nu} \times \gamma^5. \quad (S.12)$$

Problem 1(f):

$$6 i \epsilon_{\kappa\lambda\mu\nu} \gamma^0 \gamma^5 = \frac{6}{24} \gamma_\kappa \gamma^{[\kappa \gamma^\lambda \gamma^\mu \gamma^\nu]}$$

$$= \frac{1}{4} \gamma_\kappa \left( \gamma^{[\kappa \gamma^\lambda \gamma^\mu \gamma^\nu]} - \gamma^{[\lambda \gamma^\kappa \gamma^\mu \gamma^\nu]} + \gamma^{[\lambda \gamma^\mu \gamma^\kappa \gamma^\nu]} - \gamma^{[\lambda \gamma^\mu \gamma^\nu \gamma^\kappa]} \right)$$

$$= \frac{1}{4} \left( 4 \gamma^{[\lambda \gamma^\mu \gamma^\nu]} + 2 \gamma^{[\lambda \gamma^\mu \gamma^\nu]} + 4 g^{[\lambda \gamma^\mu \gamma^\nu]} + 2 \gamma^{[\nu \gamma^\mu \gamma^\lambda]} \right)$$

$$= \frac{1}{4} (4 + 2 + 0 - 2) \times \gamma^{[\lambda \gamma^\mu \gamma^\nu]} = \gamma^{[\lambda \gamma^\mu \gamma^\nu]}. \quad (S.13)$$

Problem 1(g): Proof by inspection: In the Weyl basis and $2 \times 2$ block form, the 16 matrices are

$$\gamma_4 \times 4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & +\sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

$$\frac{1}{2} \gamma^{[i \gamma^j]} = -i \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \quad \frac{1}{2} \gamma^{[0 \gamma^i]} = \begin{pmatrix} -\sigma^i & 0 \\ 0 & +\sigma^i \end{pmatrix}, \quad (S.14)$$

$$\gamma^5 \gamma^0 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}, \quad \gamma^5 \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix},$$

and their linear independence is self-evident. Since there are only 16 independent $4 \times 4$ matrices altogether, any such matrix $\Gamma$ is a linear combination of the matrices (S.14). Q.E.D. 3
**Algebraic Proof:** Without making any assumption about the matrix form of the $\gamma^\mu$ operators, let us consider the Clifford algebra (1). Using $\gamma^\mu \gamma^\nu = \pm \gamma^\nu \gamma^\mu$ where the sign is $+$ for $\mu = \nu$ and $-$ for $\mu \neq \nu$, we may re-order any product of the $\gamma$ matrices as $\pm \gamma^0 \gamma^1 \gamma^2 \gamma^3$. Moreover, since each $\gamma^\mu$ squares to $+1$ or $-1$, we may further simplify the product in question to $\pm (\gamma^0 \text{ or } 1) \times (\gamma^1 \text{ or } 1) \times (\gamma^2 \text{ or } 1) \times (\gamma^3 \text{ or } 1)$. The net result is (up to a sign or $\pm i$ factor) one of the $16$ matrices: $1$, or a $\gamma^\mu$, or a $\gamma^\mu \gamma^\nu$ for $\mu \neq \nu$ which equals to $\frac{1}{2} \gamma^{[\mu} \gamma^{\nu]}$, or a $\gamma^\lambda \gamma^\mu \gamma^\nu$ for $3$ different $\lambda, \mu, \nu$ which equals to $\frac{1}{6} \gamma^{[\lambda} \gamma^{\mu} \gamma^{\nu]} = i\epsilon^{\lambda\mu\nu\rho} \gamma^\rho$ (cf. part (f)), or $\gamma^0 \gamma^1 \gamma^2 \gamma^3 = -i\gamma^5$. Consequently, any operator $\Gamma$ algebraically constructed of the $\gamma$’s is a linear combination of these $16$ matrices.

Incidentally, this proof explains why the Dirac matrices are $4 \times 4$ in $d = 4$ spacetime dimensions: the $16$ linearly-independent products of Dirac matrices require matrix size to be $\sqrt{16} = 4$.

Technically, we may also use matrices of size $4n \times 4n$, but then we would have $\gamma^\mu = \gamma_{4 \times 4}^\mu \otimes 1_{n \times n}$, and ditto for all their products. Physically, this means combining the Dirac spinor index with some other index $i = 1, \ldots, n$ which has nothing to do with Lorentz symmetry. Nobody wants such an index confusion, so physicists always stick to $4 \times 4$ Dirac matrices in $4$ spacetime dimensions.

Problem 1**: 
As in problem 1(c), for any product $\Gamma$ of the Dirac matrices, $\gamma^\mu \Gamma = (\pm 1)^n \Gamma \gamma^\mu$ where $n$ is the number of $\gamma^{\nu \neq \mu}$ factors of $\Gamma$. For $\Gamma = \gamma^0 \gamma^1 \ldots \gamma^{d-1}$ (moduli the overall sign or $\pm i$ factor), each $\gamma^\mu$ appear once, with the remaining $d-1$ factors being $\gamma^{\nu \neq \mu}$. Thus, $n = d - 1$ and $\gamma^\mu \Gamma = (\pm 1)^{d-1} \Gamma \gamma^\mu$: for even spacetime dimensions $d$, $\Gamma$ anticommutes with all the $\gamma^\mu$ matrices, but for odd $d$, $\Gamma$ commutes with all the $\gamma^\mu$ instead of anticommuting.

Now consider the dimension of the Clifford algebra made out of Dirac matrix products and their linear combinations. As in problem 1(g), any product of Dirac matrices can be re-arranged as $\pm (\gamma^0 \text{ or } 1) \times (\gamma^1 \text{ or } 1) \times \cdots \times (\gamma^{d-1} \text{ or } 1)$, and the number of distinct products of this type is clearly $2^d$. Alternatively, we count a single unit matrix, $\binom{d}{1} = d$ of $\gamma^\mu$ matrices, $\binom{d}{2}$ of independent $\frac{1}{2} \gamma^{[\mu} \gamma^{\nu]}$ products, $\binom{d}{3}$ of $\frac{1}{6} \gamma^{[\lambda} \gamma^{\mu} \gamma^{\nu]}$ products, etc., etc., all the way up to $\binom{d}{d-1} = d$ of $\gamma^{[\beta \cdots \omega]} \propto \epsilon^{\alpha \beta \cdots \omega} \gamma^\alpha \Gamma$, and $\binom{d}{d} = 1$ of $\gamma^{[\alpha \cdots \omega]} \propto \epsilon^{\alpha \beta \cdots \omega} \Gamma$; the net number of all
such matrices is \( \sum_{k=0}^{d} \binom{d}{k} = 2^d \).

For even dimensions \( d \), all the \( 2^d \) matrix products are distinct, and we need matrices of size \( 2^{d/2} \times 2^{d/2} \) to accommodate them. But for odd \( d \), the \( \Gamma \) matrix commutes with the entire Clifford algebra (since it commutes with all the \( \gamma^\mu \)), so to avoid redundancy we should set \( \Gamma = 1 \) or \( \Gamma = -1 \). Consequently, any product of \( k \) distinct Dirac matrices becomes identical (up to a sign or \( \pm i \)) with the product of the other \( d - k \) matrices, for example \( \gamma^0 \cdots \gamma^{k-1} = i^{2k} \gamma^k \cdots \gamma^{d-1} \). This halves the net dimension of the Clifford algebra from \( 2^d \) to \( 2^{d-1} \) and calls for matrix size \( 2^{(d-1)/2} \times 2^{(d-1)/2} \).

Thus, the Dirac matrices in 2 or 3 space time dimensions are \( 2 \times 2 \), in 4 or 5 dimensions they are \( 4 \times 4 \), in 6 or 7 dimensions — \( 8 \times 8 \), in 8 or 9 dimensions — \( 16 \times 16 \), in 10 or 11 dimensions — \( 32 \times 32 \), etc., etc.

**Problem 2(a):**
I have already written down the \( \gamma^5 \) and the \( \frac{1}{2} \gamma^{[\mu} \gamma^{\nu]} = -2iS^{\mu\nu} \) in eq. (S.14), but here is the detailed calculation, in case you need it:

\[
\gamma^5 = i \begin{pmatrix} 0 & \bar{\sigma}^0 \\ \sigma^0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \bar{\sigma}^1 \\ \sigma^1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \bar{\sigma}^2 \\ \sigma^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \bar{\sigma}^3 \\ \sigma^3 & 0 \end{pmatrix}
= i \begin{pmatrix} \bar{\sigma}^0 \sigma^1 \bar{\sigma}^2 \sigma^3 & 0 \\ 0 & \sigma^0 \bar{\sigma}^1 \sigma^2 \bar{\sigma}^3 \end{pmatrix} = \begin{pmatrix} +i\sigma^1 \sigma^2 \sigma^3 & 0 \\ 0 & -i\sigma^1 \sigma^2 \sigma^3 \end{pmatrix}
= \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix},
\]

(S.15)

\[
\gamma^{\mu} \gamma^{\nu} = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & \bar{\sigma}^\nu \\ \sigma^\nu & 0 \end{pmatrix} = \begin{pmatrix} \bar{\sigma}^\mu \sigma^\nu & 0 \\ 0 & \sigma^\mu \bar{\sigma}^\nu \end{pmatrix},
\]

(S.16)

and consequently

\[
S^{ij} = \frac{i}{4} \begin{pmatrix} -\sigma^{[i} \sigma^{j]} & 0 \\ 0 & -\sigma^{[i} \sigma^{j]} \end{pmatrix} = \frac{\epsilon^{ijk}}{2} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}
\]

(S.17)

while

\[
S^{0k} = -S^{k0} = \frac{i}{2} \gamma^0 \gamma^k = \frac{i}{2} \begin{pmatrix} -\sigma^k & 0 \\ 0 & +\sigma^k \end{pmatrix}.
\]

(S.18)
Problem 2(b):
Consider a continuous Lorentz transform \( L^\mu_\nu = \exp(\Theta)^\mu_\nu \) parametrized by an antisymmetric matrix \( \Theta^{\mu\nu} = -\Theta^{\nu\mu} \). The Dirac spinor representation of this transform is
\[
M_D(L) = \exp\left(-\frac{i}{2} \Theta^{\mu\nu} S^{\mu\nu}\right).
\] (S.19)

Let’s write down the exponent here as an explicit matrix in the Weyl basis. The \( S^{\mu\nu} \) matrices are spelled out in eqs. (S.17) and (S.18), while the \( \Theta^{\mu\nu} \) are equivalent to a pair of 3-vectors \( \mathbf{a} \) and \( \mathbf{b} \) according to
\[
\Theta^{ij} = \epsilon^{ijk} a^k, \quad \Theta^{0i} = -\Theta^{i0} = b^i.
\] (S.20)

Consequently
\[
\Theta^{\mu\nu} S^{\mu\nu} = a^k \epsilon^{ijk} S^{ij} - 2b^i \times S^{0i}
\]
\[
= a^k \epsilon^{ijk} \frac{1}{2} \epsilon^{i\ell} \begin{pmatrix} \sigma^\ell & 0 \\ 0 & \sigma^\ell \end{pmatrix} - 2b^i \times \frac{i}{2} \begin{pmatrix} -\sigma^i & 0 \\ 0 & +\sigma^i \end{pmatrix}
\]
\[
= a^k \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} + b^k \begin{pmatrix} +i\sigma^k & 0 \\ 0 & -i\sigma^k \end{pmatrix}
\] (S.21)
\[
= \begin{pmatrix} (\mathbf{a} + i\mathbf{b}) \cdot \sigma & 0 \\ 0 & (\mathbf{a} - i\mathbf{b}) \cdot \sigma \end{pmatrix},
\]
and therefore
\[
M_D(L) = \exp\left(-\frac{i}{2} \Theta^{\mu\nu} S^{\mu\nu}\right)
\]
\[
= \exp\left(-\frac{i}{2} (\mathbf{a} + i\mathbf{b}) \cdot \sigma 0 \\ 0 -\frac{i}{2} (\mathbf{a} - i\mathbf{b}) \cdot \sigma \right)
\]
\[
= \begin{pmatrix} \exp\left(-\frac{i}{2} (\mathbf{a} + i\mathbf{b}) \cdot \sigma \right) & 0 \\ 0 & \exp\left(-\frac{i}{2} (\mathbf{a} - i\mathbf{b}) \cdot \sigma \right) \end{pmatrix}
\] (S.22)
\[
= \exp\left( M(L) 0 \\ 0 \overline{M}(L) \right),
\]
where \( M(L) = \exp\left(-\frac{i}{2} (\mathbf{a} + i\mathbf{b}) \cdot \sigma \right) \) and \( \overline{M}(L) = \exp\left(-\frac{i}{2} (\mathbf{a} - i\mathbf{b}) \cdot \sigma \right) \) are precisely as in
the previous homework. In other words, the $M_D(L)$ matrix is precisely as in eq. (4) with $M_L(L) \equiv M(L)$ and $M_R(L) \equiv \bar{M}(L)$.

Now consider the special cases of a pure 3D rotation or a pure Lorentz boost. For a rotation through angle $\phi$ around axis $\mathbf{n}$,

$$a = \phi \mathbf{n} \quad \text{while} \quad b = 0.$$  \hfill (S.23)

To illustrate how this works, consider a rotation around the $z$ axis. Translating the $a$ and $b$ in eq. (S.23) into the $\Theta^\mu_\nu$ matrix — with the first index \(\uparrow\) and the second index \(\downarrow\) — we get

$$\Theta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\phi & 0 \\ 0 & +\phi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies L = \exp(\Theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi & 0 \\ 0 & \sin \phi & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = R(\phi, z).$$  \hfill (S.24)

Thus, for $b = 0$ the Lorentz transform is indeed a pure 3D rotation.

Plugging $a$ and $b$ as in eq. (S.23) into the formulae for the $M = M_L$ and $\bar{M} = M_R$ matrices, we get

$$M_L = \exp\left(-\frac{i}{2}(a + ib) \cdot \sigma\right) = \exp\left(-\frac{i}{2} \phi \mathbf{n} \cdot \sigma\right),$$

$$M_R = \exp\left(-\frac{i}{2}(a - ib) \cdot \sigma\right) = \exp\left(-\frac{i}{2} \phi \mathbf{n} \cdot \sigma\right),$$  \hfill (S.25)

in perfect agreement with eq. (5).

Finally, consider a pure Lorentz boost. The $L^\mu_\nu$ matrix — with the first index \(\uparrow\) and the second index \(\downarrow\) — for a boost in $z$ direction has form

$$L = \begin{pmatrix} \gamma & 0 & 0 & \beta \gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta \gamma & 0 & 0 & \gamma \end{pmatrix} \equiv \begin{pmatrix} \cosh(r) & 0 & 0 & \sinh(r) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(r) & 0 & 0 & \cosh(r) \end{pmatrix} = \exp\left(\begin{pmatrix} 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ r & 0 & 0 & 0 \end{pmatrix}\right).$$  \hfill (S.26)
which corresponds to the $\Theta_{\mu}$ matrix

$$
\Theta = \begin{pmatrix}
0 & 0 & 0 & r \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
r & 0 & 0 & 0 \\
\end{pmatrix}.
$$

(S.27)

Similarly, for a boost in some direction $\mathbf{n}$,

$$
\Theta = \begin{pmatrix}
0 & rn^x & rn^y & rn^z \\
-\rho n^x & 0 & 0 & 0 \\
-\rho n^y & 0 & 0 & 0 \\
l n^z & 0 & 0 & 0 \\
\end{pmatrix}.
$$

(S.28)

Note that this matrix is for the $\Theta_{\mu}$ with the first index $\uparrow$ and the second index $\downarrow$; when both indices are $\uparrow$ we get

$$
\Theta^{k0} = +rn^k, \quad \Theta^{0k} = -rn^k, \quad \Theta_{ij} = 0.
$$

(S.29)

In terms of the $\mathbf{a}$ and $\mathbf{b}$ vectors, this means

$$
\mathbf{a} = 0 \quad \text{while} \quad \mathbf{b} = -rn,
$$

(S.30)

hence

$$
M_L = \exp(-\frac{i}{2}(\mathbf{a} + i\mathbf{b}) \cdot \mathbf{\sigma}) = \exp(-\frac{1}{2}rn \cdot \mathbf{\sigma}),
$$

$$
M_R = \exp(-\frac{i}{2}(\mathbf{a} - i\mathbf{b}) \cdot \mathbf{\sigma}) = \exp(+\frac{1}{2}rn \cdot \mathbf{\sigma}),
$$

(S.31)

in perfect agreement with eq. (6).

To rewrite these formulae in terms of the $\beta$ and $\gamma$ parameters of the Lorentz boost, note
that \((\mathbf{n} \cdot \sigma)^2 = 1\), which leads to

\[
M_L^2 = \exp(-r \mathbf{n} \cdot \sigma) = \sum_{k=0}^{\infty} \frac{(-1)^k r^k}{k!} \times (\mathbf{n} \cdot \sigma)^k
\]

\[
= \sum_{\text{even } k} \frac{r^k}{k!} \times 1 - \sum_{\text{odd } k} \frac{r^k}{k!} \times (\mathbf{n} \cdot \sigma)
= \cosh(r) \times 1 - \sinh(r) \times (\mathbf{n} \cdot \sigma)
= \gamma \times 1 - \beta \gamma (\mathbf{n} \cdot \sigma)
\]

while \(M_R^2 = \exp(+r \mathbf{n} \cdot \sigma) = \gamma \times 1 + \beta \gamma (\mathbf{n} \cdot \sigma)\).

Consequently,

\[
M_L = \sqrt{\gamma - \beta \gamma \mathbf{n} \cdot \sigma} \quad \text{and} \quad M_R = \sqrt{\gamma + \beta \gamma \mathbf{n} \cdot \sigma}.
\]

Problem 2(c):

\[
\psi'_L(x') = M_L \times \psi_L(x), \\
\psi'^*_L(x') = M_L^* \times \psi_L(x),
\]

\[
\sigma_2 \times \psi'^*_L(x') = \sigma_2 \times M_L^* \times \psi^*_L(x) = \sigma_2 M_L^* \sigma_2 \times \sigma_2 \psi^*_L(x) = M_R \times \sigma_2 \psi^*_L(x),
\]

thus \(\sigma_2 \times \psi^*_L(x)\) transforms under the continuous Lorentz transforms exactly like the \(\psi_R(x)\).

Likewise,

\[
\psi'_R(x') = M_R \times \psi_R(x), \\
\psi'^*_R(x') = M_R^* \times \psi_R(x),
\]

\[
\sigma_2 \times \psi'^*_R(x') = \sigma_2 \times M_R^* \times \psi^*_R(x) = \sigma_2 M_R^* \sigma_2 \times \sigma_2 \psi^*_R(x) = M_L \times \sigma_2 \psi^*_R(x),
\]

thus \(\sigma_2 \times \psi^*_R(x)\) transforms under the continuous Lorentz transforms exactly like the \(\psi_L(x)\).
Problem 2(d–e):

Given the $\gamma^\mu$ matrices (3) and the decomposition (4) of the Dirac spinor field $\Psi(x)$ into 2 Weyl spinor fields $\psi_L(x)$ and $\psi_R(x)$, we have

$$(i\gamma^\mu \partial_\mu - m)\Psi = \begin{pmatrix} -m & i\bar{\sigma}^\mu \partial_\mu \\ i\sigma^\mu \partial_\mu & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} -m \psi_L + i\bar{\sigma}^\mu \partial_\mu \psi_R \\ i\sigma^\mu \partial_\mu \psi_L - m \psi_R \end{pmatrix} \quad (S.35)$$

while

$$\bar{\Psi} = \Psi^\dagger \gamma^0 = \begin{pmatrix} \psi_L^\dagger & \psi_R^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \psi_L^\dagger & \psi_R^\dagger \end{pmatrix}. \quad (S.36)$$

Consequently, the Dirac Lagrangian becomes

$$\mathcal{L} = \bar{\Psi} (i\gamma^\mu \partial_\mu - m)\Psi$$

$$= \psi_R^\dagger \left( -m \psi_L + i\bar{\sigma}^\mu \partial_\mu \psi_R \right) + \psi_L^\dagger \left( i\sigma^\mu \partial_\mu \psi_L - m \psi_R \right) \quad (S.37)$$

$$= i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R - m \psi_L^\dagger \psi_R - m \psi_R^\dagger \psi_L.$$

For massless fermions, the last two terms on the last line here go away and we are left with

$$\mathcal{L}_{\text{massless}} = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R \quad (S.38)$$

where the two Weyl spinor fields $\psi_L(x)$ and $\psi_R(x)$ are completely independent from each other. In particular, they obey independent Weyl equations

$$i\bar{\sigma}^\mu \partial_\mu \psi_L = 0 \quad \text{and} \quad i\sigma^\mu \partial_\mu \psi_R = 0. \quad (S.39)$$

On the other hand, for $m \neq 0$ the two last terms in the Lagrangian (S.37) connect the $\psi_L$ and $\psi_R$ to each other and we cannot have one without the other. In particular, their equations of motion become mixed,

$$i\bar{\sigma}^\mu \partial_\mu \psi_L = m \psi_R \quad \text{and} \quad i\sigma^\mu \partial_\mu \psi_R = m \psi_L. \quad (S.40)$$
Problem 3(a):
For \( p = 0 \), \( p^0 = +m \) and \( p - m = m(\gamma^0 - 1) \). Hence, \( u(p = 0, s) \) satisfy \((\gamma^0 - 1)u = 0\), or in the Weyl basis
\[
\begin{pmatrix}
-1_{2 \times 2} & 1_{2 \times 2} \\
1_{2 \times 2} & -1_{2 \times 2}
\end{pmatrix}
\begin{pmatrix}
u
\end{pmatrix}
= 0 \implies \begin{pmatrix}
u
\end{pmatrix}
= \begin{pmatrix}
\zeta \\
\zeta
\end{pmatrix}
\]
(S.41)
where \( \zeta \) is an arbitrary two-component spinor. It’s normalization follows from \( u^\dagger u = 2 \zeta^\dagger \zeta \), so if we want \( u^\dagger u = 2 \sqrt{m} \zeta \) (for \( p = 0 \)) we need \( \zeta^\dagger \zeta = m \). Equivalently, we want \( \zeta = \sqrt{m} \xi \) — and hence \( u \) as in eq. (11) — for a conventionally normalized spinor \( \xi \), \( \xi^\dagger \xi = 0 \).

Note that there are two independent choices of \( \xi \), normalized to \( \xi^\dagger \xi = \delta_{s,s'} \) so that \( u^\dagger(0,s)u(0,s') = 2m \delta_{s,s'} = 2E_p \delta_{s,s'} \). They correspond to two spin states of the \( p = 0 \) electron. In terms of the spin vector, \( S = \frac{1}{2} \xi^\dagger \sigma \xi \).

Problem 3(b):
The Dirac equation is Lorentz-covariant, so we may obtain solutions for all \( p^\mu = (+E, p) \) by simply Lorentz-boosting the solutions (11) from the rest frame where \( p^0_0 = (+m, 0) \). Thus,
\[
u(p, s) = MD(B)u(p_0, s) = \begin{pmatrix}
M_L & 0 \\
0 & M_R
\end{pmatrix}
\begin{pmatrix}
\sqrt{m} \xi_s \\
\sqrt{m} \xi_s
\end{pmatrix}
= \begin{pmatrix}
\sqrt{m} M_L \xi_s \\
\sqrt{m} M_R \xi_s
\end{pmatrix}
\]
(S.42)
where \( M_D, M_L, \) and \( M_R \) are respectively Dirac-spinor, LH–Weyl-spinor, and RH–Weyl-spinor representations or the Lorentz boost \( B \) from \( p^\mu = (m, 0) \) to \( p^\mu = (E, p) \). Specifically, this boost has
\[
\gamma = \frac{E}{m}, \quad \gamma \beta n = \frac{p}{m}
\]
(S.43)
so eqs. (7) for its Weyl-spinor representations \( M_L \) and \( M_R \) may be written as
\[
\sqrt{m} M_L = \sqrt{E - p \cdot \sigma}, \quad \sqrt{m} M_R = \sqrt{E + p \cdot \sigma}.
\]
(S.44)
Plugging in these formulae into eq. (S.42) immediately gives us
\[
u(p, s) = \begin{pmatrix}
\sqrt{E - p \cdot \sigma} \xi_s \\
\sqrt{E + p \cdot \sigma} \xi_s
\end{pmatrix}
\]
(S.45)
in accordance with eq. (12).
Problem 3(c):
The negative-frequency solutions $e^{+ipx}v(p, s)$ have Dirac spinors $v$ satisfying $(p + m)v = 0$. For a particle at rest, $p^\mu = (+m, 0)$, this equation becomes $m(\gamma^0 + 1)v = 0$, or in the Weyl basis

$$\begin{pmatrix} 1_{2\times 2} & 1_{2\times 2} \\ 1_{2\times 2} & 1_{2\times 2} \end{pmatrix} v(p = 0, s) = 0 \implies v(p = 0, s) = \sqrt{m} \begin{pmatrix} +\eta_s \\ -\eta_s \end{pmatrix} \quad (S.46)$$

for some two-component spinor $\eta_s$. As in part (a), the $\sqrt{m}$ factor translates the normalization $v^\dagger(p, s)v(p, s') = 2E_p\delta_{s,s'} = 2m\delta_{s,s'}$ (for $p = 0$) to $\eta_s^\dagger\eta_{s'} = \delta_{s,s'}$.

For $p \neq 0$ we proceed similarly to part (b), namely Lorentz-boost the rest-frame solution (S.46) to the frame where $p^\mu = (+E_p, p)$:

$$v(p, s) = MD(B)v(p_0, s) = \begin{pmatrix} M_L & 0 \\ 0 & M_R \end{pmatrix} \begin{pmatrix} +\sqrt{m}\eta_s \\ -\sqrt{m}\eta_s \end{pmatrix} = \begin{pmatrix} +\sqrt{E - p \cdot \sigma}\eta_s \\ -\sqrt{E + p \cdot \sigma}\eta_s \end{pmatrix} \quad (S.47)$$

in accordance with eq. (13)

Problem 3(d):
Physically, a hole in the Fermi sea has opposite free energy, opposite momentum, opposite spin, etc., from the missing fermion (see [my notes] for the explanation). A positron is a hole in the Dirac sea of electrons, so it should have opposite $p^\mu = (E, p)$ from the missing electron state — that’s why the $v(p, s)$ spinors accompany the $e^{+ipx} = e^{+iEt - ipx}$ plane waves instead of the $e^{-ipx} = E^{-iEt + ipx}$ factors of the $u(p, s)$ spinors. Likewise, the positron should have the opposite spin state from the missing electron, and that’s why the $\eta_s$ should have the opposite spin from the $\xi_s$. More accurately, the $\eta_s$ should carry the opposite spin vector $\eta_s^\dagger S\eta_s$ from the $\xi_s^\dagger S\xi_s$.

The solution to this spin relation is $\eta_s = \sigma_2\xi_s^*$ (where * denotes complex conjugation). Indeed, using the Pauli matrix relation $\sigma_2\sigma_2^* = -\sigma$ from the [homework set #10], we see that

$$\eta = \sigma_2\xi^* \implies \eta^\dagger = \xi^\top\sigma_2$$
\[ \eta^\dagger\sigma\eta = \xi^\dagger\sigma\sigma_2\xi^* = (\xi^\dagger\sigma_2\sigma^*\sigma_2\xi^*)^* = (-\xi^\dagger\sigma\xi)^* = -\xi^\dagger\sigma\xi \]

\[ \eta^\dagger S\eta = \frac{1}{2} \eta^\dagger\sigma\eta = -\xi^\dagger S\xi. \quad (S.48) \]

Also, for \( \eta_s = \sigma_2\xi_s^* \) we have

\[ v(p, s) = \begin{pmatrix} +\sqrt{E - p \cdot \sigma} \eta_s \\ -\sqrt{E + p \cdot \sigma} \eta_s \end{pmatrix} = \begin{pmatrix} +\sqrt{E - p \cdot \sigma} \sigma_2\xi_s^* \\ -\sqrt{E + p \cdot \sigma} \sigma_2\xi_s^* \end{pmatrix} \]

\[ \begin{pmatrix} +\sigma_2^2\sqrt{E - p \cdot \sigma} \sigma_2\xi_s^* \\ -\sigma_2^2\sqrt{E + p \cdot \sigma} \sigma_2\xi_s^* \end{pmatrix} \]

\[ \begin{pmatrix} 0 & +\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} +\sqrt{E - p \cdot \sigma} \xi_s \\ +\sqrt{E + p \cdot \sigma} \xi_s \end{pmatrix}^* \]

\[ = \gamma^2 u^*(p, s). \]

Similarly, \( u(p, s) = \gamma^2 v^*(p, s) \). Indeed,

\[ \gamma^2 v^*(p, s) = \gamma^2 (\gamma^2 u^*(p, s))^* = \gamma^2 (\gamma^2)^* u(p, s) = u(p, s) \quad (S.50) \]

since \( \gamma^2 (\gamma^2)^* = \gamma^2 (-\gamma^2) = +1. \)

Problem 3(e):

The 3D spinors \( \xi_\lambda \) of definite helicity \( \lambda = \mp \frac{1}{2} \) satisfy

\[ (p \cdot \sigma)\xi_\mp = \mp|p| \times \xi_\mp. \quad (S.51) \]

Plugging these \( \xi_\lambda \) into the positive-energy Dirac spinors (14), we obtain

\[ u(p, \lambda = \mp \frac{1}{2}) = \begin{pmatrix} \sqrt{E \mp |p|} \times \xi_\mp \\ \sqrt{E \mp |p|} \times \xi_\mp \end{pmatrix}. \quad (S.52) \]

In the ultra-relativistic limit \( E \approx |p| \gg m \), the square roots here simplify to \( \sqrt{E \mp |p|} \approx \sqrt{2}E \).
and $\sqrt{E - |p|} \approx 0$ (by comparison with the other root). Consequently, eq. (S.52) simplifies to
\[
\begin{align*}
u(p, L) & \approx \sqrt{2E} \begin{pmatrix} \xi_L \\ 0 \end{pmatrix}, & \nu(p, R) & \approx \sqrt{2E} \begin{pmatrix} 0 \\ \xi_R \end{pmatrix}.
\end{align*}
\] (S.53)

In other words, the ultra-relativistic positive-energy Dirac spinors of definite helicity are chiral — dominated by the LH Weyl components for the left helicity or by the RH Weyl components for the right helicity.

Now consider the negative-energy Dirac spinors (15). Because the $\eta_s$ spinors have exactly opposite spins from the $\xi_s$, their helicities are also opposite and hence
\[
(p \cdot \sigma)\eta_{\mp} = \pm |p| \times \eta_{\mp}
\] (S.54)
— note the opposite sign from eq. (S.51). Therefore, the negative-energy Dirac spinors $v$ of definite helicity are
\[
v(p, \lambda = \mp \frac{1}{2}) = \begin{pmatrix} +\sqrt{E \mp |p|} \times \eta_{\mp} \\ -\sqrt{E \pm |p|} \times \eta_{\mp} \end{pmatrix},
\] (S.55)
and in the ultra-relativistic limit they become
\[
v(p, L) \approx -\sqrt{2E} \begin{pmatrix} 0 \\ \eta_L \end{pmatrix}, & \nu(p, R) & \approx +\sqrt{2E} \begin{pmatrix} \eta_R \\ 0 \end{pmatrix}.
\] (S.56)

Again, the ultra-relativistic negative-energy spinors are chiral, but this time the chirality is opposite from the helicity — the left-helicity spinor has dominant RH Weyl components while the right-helicity spinor has dominant LH Weyl components

Problem 4(a):  
First, let’s check the normalization conditions for the Dirac spinors (12) and (13) for general moments:
\[
\begin{align*}
u^\dagger(p, s)u(p, s') & = \xi_s^\dagger \left( \left( \sqrt{E - p \cdot \sigma} \right)^2 + \left( \sqrt{E + p \cdot \sigma} \right)^2 \right) \xi_{s'} = \xi_s^\dagger (2E) \xi_{s'} = 2E \delta_{s,s'}, \\
v^\dagger(p, s)v(p, s') & = \eta_s^\dagger \left( \left( +\sqrt{E - p \cdot \sigma} \right)^2 + \left( -\sqrt{E + p \cdot \sigma} \right)^2 \right) \eta_{s'} = \eta_s^\dagger (2E) \eta_{s'} = 2E \delta_{s,s'}.
\end{align*}
\] (S.57)
Note that in the second equation $\eta_s^\dagger \eta_{s'} = \xi_s^\dagger \sigma_2 \sigma_2 \xi_{s'} = \left( \xi_s \xi_{s'} \right)^* = \delta_{s,s'}$. 

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Now consider the Lorentz invariant products $\bar{u}u$ and $\bar{v}v$. The $\bar{u}$ and $\bar{v}$ are given by

$$u(p, s) = u^\dagger(p, s)\gamma^0 = \left(\frac{\sqrt{E - \mathbf{p} \cdot \sigma}}{\sqrt{E + \mathbf{p} \cdot \sigma}}\right)\xi_s^\dagger\left(\begin{array}{cc} 0 & 1_{2\times2} \\ 1_{2\times2} & 0 \end{array}\right)$$

$$= (\xi_s^\dagger \times \sqrt{E + \mathbf{p} \cdot \sigma}, \xi_s^\dagger \times \sqrt{E - \mathbf{p} \cdot \sigma}),$$

$$v(p, s) = v^\dagger(p, s)\gamma^0 = \left(\frac{\sqrt{E - \mathbf{p} \cdot \sigma}}{\sqrt{E + \mathbf{p} \cdot \sigma}}\right)\eta_s^\dagger\left(\begin{array}{cc} 0 & 1_{2\times2} \\ 1_{2\times2} & 0 \end{array}\right)$$

$$= (-\eta_s^\dagger \times \sqrt{E + \mathbf{p} \cdot \sigma}, +\eta_s^\dagger \times \sqrt{E - \mathbf{p} \cdot \sigma}).$$

Consequently,

$$\bar{u}(p, s) u(p, s') = \xi_s^\dagger \times \sqrt{E + \mathbf{p} \cdot \sigma} \times \sqrt{E - \mathbf{p} \cdot \sigma} \times \xi_{s'}$$

$$+ \xi_s^\dagger \times \sqrt{E - \mathbf{p} \cdot \sigma} \times \sqrt{E + \mathbf{p} \cdot \sigma} \times \xi_{s'}$$

$$= 2m \times \eta_s^\dagger \eta_{s'} = 2m \delta_{s, s'}.$$

because

$$\sqrt{E + \mathbf{p} \cdot \sigma} \times \sqrt{E - \mathbf{p} \cdot \sigma} = \sqrt{E - \mathbf{p} \cdot \sigma} \times \sqrt{E + \mathbf{p} \cdot \sigma}$$

$$= \sqrt{E^2 - (\mathbf{p} \cdot \sigma)^2} = \sqrt{E^2 - \mathbf{p}^2} = m.$$

Likewise,

$$\bar{v}(p, s) v(p, s') = -\eta_s^\dagger \times \sqrt{E + \mathbf{p} \cdot \sigma} \times \sqrt{E - \mathbf{p} \cdot \sigma} \times \eta_{s'}$$

$$- \eta_s^\dagger \times \sqrt{E - \mathbf{p} \cdot \sigma} \times \sqrt{E + \mathbf{p} \cdot \sigma} \times \eta_{s'}$$

$$= -2m \times \eta_s^\dagger \eta_{s'} = -2m \delta_{s, s'}.$$

Problem 4(b):

In matrix notations (column $\times$ row = matrix), we have

$$u(p, s) \times \bar{u}(p, s) = \left(\frac{\sqrt{E - \mathbf{p} \sigma}}{\sqrt{E + \mathbf{p} \sigma}}\right)\xi_s^\dagger \left(\begin{array}{cc} \xi_s \sqrt{E + \mathbf{p} \sigma} & \xi_s \sqrt{E - \mathbf{p} \sigma} \\ \xi_s \sqrt{E + \mathbf{p} \sigma} & \xi_s \sqrt{E - \mathbf{p} \sigma} \end{array}\right)$$

$$= \left(\begin{array}{cc} \sqrt{E - \mathbf{p} \sigma} \xi_s \sqrt{E + \mathbf{p} \sigma} & \sqrt{E - \mathbf{p} \sigma} \xi_s \sqrt{E - \mathbf{p} \sigma} \\ \sqrt{E + \mathbf{p} \sigma} \xi_s \sqrt{E + \mathbf{p} \sigma} & \sqrt{E + \mathbf{p} \sigma} \xi_s \sqrt{E - \mathbf{p} \sigma} \end{array}\right).$$
\[
\begin{align*}
v(p,s) \times \overline{v}(p,s) &= \left(\frac{\sqrt{E - p\sigma} \eta_s}{-\sqrt{E + p\sigma} \eta_s}\right) \times \left(\eta_s^\dagger \sqrt{E + p\sigma}, + \eta_s^\dagger \sqrt{E - p\sigma}\right) \\
&= \left(\frac{-\sqrt{E - p\sigma} (\eta_s \times \eta_s^\dagger) \sqrt{E + p\sigma} + \sqrt{E - p\sigma} (\eta_s \times \eta_s^\dagger) \sqrt{E - p\sigma}}{+\sqrt{E + p\sigma} (\eta_s \times \eta_s^\dagger) \sqrt{E + p\sigma} - \sqrt{E + p\sigma} (\eta_s \times \eta_s^\dagger) \sqrt{E - p\sigma}}\right). 
\end{align*}
\]

Summing over two spin polarizations replaces \(\xi_s \times \xi_s^\dagger\) with \(\sum_s \xi_s \times \xi_s^\dagger = 1_{2 \times 2}\) and likewise \(\eta_s \times \eta_s^\dagger\) with \(\sum_s \eta_s \times \eta_s^\dagger = 1_{2 \times 2}\). Consequently,

\[
\sum_s u(p,s) \times \overline{u}(p,s) = \left(\begin{array}{cc}
\sqrt{E - p\sigma} [\sum_s \xi_s \times \xi_s^\dagger] \sqrt{E + p\sigma} & \sqrt{E - p\sigma} [\sum_s \xi_s \times \xi_s^\dagger] \sqrt{E - p\sigma} \\
\sqrt{E + p\sigma} [\sum_s \xi_s \times \xi_s^\dagger] \sqrt{E + p\sigma} & \sqrt{E + p\sigma} [\sum_s \xi_s \times \xi_s^\dagger] \sqrt{E - p\sigma}
\end{array}\right).
\]

\[
\sum_s v(p,s) \times \overline{v}(p,s) = \left(\begin{array}{cc}
-m (E - p\sigma) & m (E - p\sigma) \\
E + p\sigma & m (E - p\sigma)
\end{array}\right) = m \times 1_{4 \times 4} + E \times \gamma^0 - p \cdot \vec{\gamma}
\]

\[
\sum_s v(p,s) \times \overline{v}(p,s) = \left(\begin{array}{cc}
-m (E - p\sigma) & m (E - p\sigma) \\
E + p\sigma & m (E - p\sigma)
\end{array}\right) = m \times 1_{4 \times 4} + E \times \gamma^0 - p \cdot \vec{\gamma}
\]

Q.E.D.