Problem 1(a):
The fermionic Fock space of Dirac fields $\hat{\Psi}(x)$ and $\hat{\bar{\Psi}}(x)$ spans states $|F_1, \ldots, F_m, F'_1, \ldots, F'_n\rangle$ (short hand for $|F(p_1, s_1), \ldots, F(p_m, s_m); F'(p'_1, s'_1), \ldots, F'(p'_n, s'_n)\rangle$) with arbitrary numbers of fermions $F$ and/or antifermions $\bar{F}$. When the $\hat{C}$ operator acts on such a multi-particles states, each particle turns to its anti-particle according to eqs. (2), thus

$$
\hat{C}|F_1, \ldots, F_m, F'_1, \ldots, F'_n\rangle = |F_1, \ldots, F_m, F'_1, \ldots, F'_n\rangle.
$$

(S.1)

Now compare the action of $\hat{C}$ on any particular state $|F_1, \ldots, F_m, F'_1, \ldots, F'_n\rangle$ and on the state $|F_1, \ldots, F_m, F'_1, \ldots, F'_n, F(p, s)\rangle = \hat{a}^\dagger_{p, s}|F_1, \ldots, F_m, F'_1, \ldots, F'_n\rangle$ (S.2) with an extra fermion:

$$
\hat{C} \times \hat{a}^\dagger_{p, s}|F_1, \ldots, F_m, F'_1, \ldots, F'_n\rangle = \hat{C}|F_1, \ldots, F_m, F'_1, \ldots, F'_n, F(p, s)\rangle
$$

$$
= |F_1, \ldots, F_m, F'_1, \ldots, F'_n, F(p, s)\rangle
$$

$$
= \hat{b}^\dagger_{p, s} \times |F_1, \ldots, F_m, F'_1, \ldots, F'_n\rangle
$$

$$
= \hat{b}^\dagger_{p, s} \times \hat{C}|F_1, \ldots, F_m, F'_1, \ldots, F'_n\rangle.
$$

(S.3)

Since this relation holds for any multi-fermion state $|F_1, \ldots, F_m, F'_1, \ldots, F'_n\rangle$ it implies oper-atorial identity

$$
\hat{C} \times \hat{a}^\dagger_{p, s} = \hat{b}^\dagger_{p, s} \times \hat{C}.
$$

(S.4)

Furthermore, since $\hat{C}^2 = 1$, this formula leads to

$$
\hat{C} \times \hat{a}^\dagger_{p, s} \times \hat{C} = \hat{b}^\dagger_{p, s} \times \hat{C} \times \hat{C} = \hat{b}^\dagger_{p, s}
$$

(S.5)

and also

$$
\hat{C} \times \hat{b}^\dagger_{p, s} \times \hat{C} = \hat{C} \times (\hat{b}^\dagger_{p, s} \times \hat{C} = \hat{C} \times \hat{a}^\dagger_{p, s}) = \hat{a}^\dagger_{p, s}.
$$

(S.6)

This proves two of the equations (2); the other two equations follow by hermitian conjugation:
since \( \hat{C} \dagger = \hat{C} \),
\[
\hat{C} \hat{a}_{p,s} \hat{C} = (\hat{C} \hat{a}_{p,s} \hat{C})^\dagger = (\hat{b}_{p,s})^\dagger = \hat{b}_{p,s},
\]
\[
\hat{C} \hat{b}_{p,s} \hat{C} = (\hat{C} \hat{b}_{p,s} \hat{C})^\dagger = (\hat{a}_{p,s})^\dagger = \hat{a}_{p,s}.
\]

(S.7)

Problem 1(b):
The Dirac fields decompose into creation and annihilation operators as
\[
\hat{\Psi}(x) = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left( e^{-ipx} u(p,s) \hat{a}_{p,s} + e^{+ipx} v(p,s) \hat{b}_{p,s} \right) \delta_p = +E_p,
\]
\[
\hat{\Psi}(x) = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left( e^{+ipx} \bar{u}(p,s) \hat{b}_{p,s}^\dagger + e^{-ipx} \bar{v}(p,s) \hat{a}_{p,s}^\dagger \right) \delta_p = +E_p,
\]

(S.8)

where \( e^{-ipx} u(p,s) \) and \( e^{+ipx} v(p,s) \) are plane-wave solutions of the Dirac equation normalized to
\[
\bar{u}^\dagger(p,s) u(p,s') = \bar{v}^\dagger(p,s) v(p,s') = 2E_p \delta_{s,s'}.
\]

(S.9)

The spinors \( u(p,s) \) and \( v(p,s) \) were explained in detail in the homework set #7 (problems 3 and 4). In particular, in problem 3(d) we saw that for any on-shell momentum \( p^\mu = (+E_p, \mathbf{p}) \) and any spin \( s \), the \( u(p,s) \) and \( v(p,s) \) spinors are related as \( v(p,s) = \gamma^2 u^*(p,s) \) (in the Weyl basis). Likewise, \( u(p,s) = \gamma^2 v^*(p,s) \); indeed,
\[
\gamma^2 v^*(p,s) = \gamma^2 \left( \gamma^2 u^*(p,s) \right)^* = \gamma^2 \gamma^2 u(p,s) = u(p,s)
\]

(S.11)

since \( \gamma^{2*} = -\gamma^2 \) and \( \gamma^2 \gamma^2 = -1 \).

After all these preliminaries, let’s sandwich the Dirac field \( \hat{\Psi}(x) \) between two \( \hat{C} \) operators. Applying eqs. (2) to each annihilation or creation operators in the expansion (S.8), we obtain
\[
\hat{C} \hat{\Psi}(x) \hat{C} = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left( e^{-ipx} u(p,s) \times \hat{C} \hat{a}_{p,s} \hat{C} + e^{+ipx} v(p,s) \times \hat{C} \hat{b}_{p,s} \hat{C} \right) \delta_p = +E_p
\]
\[
= \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left( e^{-ipx} u(p,s) \times \hat{b}_{p,s} + e^{+ipx} v(p,s) \times \hat{a}_{p,s}^\dagger \right) \delta_p = +E_p
\]

(S.12)
4-component Dirac spinors from column-vectors to row vectors, thus plex conjugation of the numerical factors accompanying them — but without transposing
The \( u \) and the Dirac conjugates of the spinor relations
\[
\hat{\Psi}(x) = \begin{pmatrix}
\psi_1^\dagger(x) \\
\psi_2^\dagger(x) \\
\psi_3^\dagger(x) \\
\psi_4^\dagger(x)
\end{pmatrix}
\]
while \( \hat{\Psi}'(x) = \begin{pmatrix}
\psi_1^\dagger(x), \psi_2^\dagger(x), \psi_3^\dagger(x), \psi_4^\dagger(x)
\end{pmatrix} \).

The charge conjugation of the \( \hat{\Psi} \) field works similarly to the \( \hat{\Psi} \) field. Again, we use eqs. (2) and the Dirac conjugates of the spinor relations \( u = \gamma^2 v^*, \ v = \gamma^2 u^* \) (for the same \((p,s)\)) — \( \bar{u} = \bar{v}^* \gamma^2 \) and \( \bar{v} = \bar{u}^* \gamma^2 \). Thus,
\[
\hat{\Psi}^\dagger(x) \hat{\Psi} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left( e^{+ip \cdot x} \bar{u}_p(s) \times \hat{\bar{\gamma}}^\dagger \hat{\bar{C}} + e^{-ip \cdot x} \bar{v}(p,s) \times \hat{\bar{\gamma}} \hat{\bar{C}} b_{p,s} \hat{\bar{C}} \right)_{p_0=+E_p} \\
= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left( e^{+ip \cdot x} \bar{u}_p(s) \times \hat{\bar{b}}^\dagger \hat{\bar{C}} + e^{-ip \cdot x} \bar{v}(p,s) \times \hat{\bar{a}} \hat{\bar{C}} \right)_{p_0=+E_p} \\
= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left( e^{+ip \cdot x} \bar{u}_p(s) \gamma^2 \times \hat{\bar{b}}^\dagger \hat{\bar{C}} + e^{-ip \cdot x} \bar{v}^*(p,s) \gamma^2 \times \hat{\bar{a}} \hat{\bar{C}} \right)_{p_0=+E_p} \\
= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left( (e^{-ip \cdot x} \bar{v}(p,s))^* \times \hat{\bar{b}}^\dagger \hat{\bar{C}} + (e^{+ip \cdot x} \bar{u}_p(s))^* \times \hat{\bar{a}} \hat{\bar{C}} \right)_{p_0=+E_p} \times \gamma^2 \\
= \hat{\Psi} \times \gamma^2.
\]
**Problem 1(c):**
First, we need a **lemma:** In the Weyl basis, all 4 Dirac matrices $\gamma^\mu$ satisfy $\gamma^\mu \gamma^2 = -\gamma^2 (\gamma^\mu)^*$.  

**Proof:** In the Weyl basis, the $\gamma^2$ matrix is imaginary while the other 3 Dirac matrices $\gamma^0, \gamma^1,$ and $\gamma^3$ are real. Also, the $\gamma^2$ commutes with itself while the other Dirac matrices anticommute with the $\gamma^2$. Hence, 

\[
\begin{align*}
\text{for } \mu = 2, \quad & \gamma^\mu \gamma^2 = +\gamma^2 \gamma^\mu = -\gamma^2 (\gamma^\mu)^*, \\
\text{for } \mu \neq 2, \quad & \gamma^\mu \gamma^2 = -\gamma^2 \gamma^\mu = -\gamma^2 (\gamma^\mu)^*.
\end{align*}
\] (S.16)

Thanks to this lemma, Dirac’s differential operator $D = (i \partial - m) = (i \gamma^\mu \partial_\mu - m)$ satisfies $D \gamma^2 = + \gamma^2 D^*$. Indeed,  

\[
(i \gamma^\mu \partial_\mu - m) \gamma^2 = i (\gamma^\mu \gamma^2) \partial_\mu - m \gamma^2 = -i \gamma^2 \gamma^\mu^* \partial_\mu \gamma^2 m = \gamma^2 (i \gamma^\mu \partial_\mu - m)^*. \quad (S.17)
\]

Now, suppose the spinor field $\Psi(x)$ satisfies the Dirac equation $(i \partial - m) \Psi = 0$. Then the charge-conjugate field $\Psi'(x) = \gamma^2 \Psi^*(x)$ also satisfies the Dirac equation:  

\[
(i \partial - m) \Psi' = (i \partial - m) (\gamma^2) \Psi^* = \gamma^2 (i \partial - m)^* \Psi^* = \gamma^2 [(i \partial - m) \Psi]^* = 0. \quad (S.18)
\]

Moreover, the $(i \partial - m) \Psi$ transforms under charge conjugation exactly like the Dirac spinor field itself. In other words, the Dirac equation transforms **covariantly.**

**Problem 1(d):**
Earlier in this problem we saw that under charge conjugation  

\[
\hat{C} \Psi \hat{C} = \gamma^2 \Psi, \quad \hat{C} \Psi^2 \hat{C} = \Psi^* \gamma^2, \quad \text{and} \quad \hat{C} (i \partial - m) \Psi \hat{C} = \gamma^2 (i \partial - m)^* \Psi^*. \quad (S.19)
\]

Consequently, the Dirac Lagrangian transforms as  

\[
\hat{C} \mathcal{L} \hat{C} = \hat{C} \overline{\Psi} (i \partial - m) \Psi \hat{C} = + \overline{\Psi} \gamma^2 \times \gamma^2 (i \partial - m)^* \Psi = - \overline{\Psi} (i \partial - m)^* \Psi^* \quad \text{naively} = - \mathcal{L}^*. \quad (S.20)
\]

Since we expect the Dirac action — and hence the Lagrangian $\mathcal{L}$ — to be real, it seems to flip sign under charge conjugation instead of being invariant. However, the complex conjugation of the classical fermionic fields and their products is tricky because their values are anti-commuting Grassmann numbers instead of the ordinary real or complex numbers.
In class, I have not discussed the mathematics of Grassmann numbers in much detail, since all we really need to know is that the odd GN anticommute with each other while the even GN commute with each other and with the odd GN. Also, the GN-valued classical fermionic fields $\Psi(x)$ must make sense as the classical limits of the operator-valued quantum fields $\hat{\Psi}(x)$. Thus, regardless of the precise definition of the complex conjugation for the Grassmann numbers, we know that it should work similarly to the hermitian conjugation of operators in the Fock space. In particular, for any GN $g$, $(g^*)^* = g$, $(cg)^* = c^*g^*$ for any c-number $c$, and for any 2 GNs $g_1$ and $g_2$, $(g_1g_2)^* = g_2^*g_1^*$ — under conjugation, the order of the product is reversed, similar to $(\hat{A}_1\hat{A}_2)^\dagger = \hat{A}_2^\dagger\hat{A}_1^\dagger$ for any operators $\hat{A}_1$ and $\hat{A}_2$. For two odd GN such as classical fermionic fields $F_1$ and $F_2$, this means

$$
(F_1 \times F_2)^* = F_2^* \times F_1^* = -F_1^* \times F_2^*. 
$$

(S.21)

In particular, in eq. (S.20)

$$
\bar{\Psi}^* \times (i\not{\partial} - m)^* \Psi^* = - (\bar{\Psi} \times (i\not{\partial} - m) \Psi)^*
$$

(S.22)

and hence

$$
\hat{C} \mathcal{L} \hat{C} = +\mathcal{L}^*.
$$

(S.23)

Since the Dirac Lagrangian is real up to a total derivative, this makes the Dirac action invariant under charge conjugation $\hat{C}$.

Alternative proof:
In problem 4(d–e) of this homework we shall see that

$$
\hat{C} \bar{\Psi} \Psi \hat{C} = +\bar{\Psi} \Psi \quad \text{while} \quad \hat{C} \bar{\Psi} \gamma^\mu \Psi \hat{C} = -\bar{\Psi} \gamma^\mu \Psi.
$$

(S.24)

The first formula here provides for C-invariance of the mass term $-m\bar{\Psi} \Psi$ in the Dirac Lagrangian, while the second formula can be generalized to 2 different Dirac spinors transforming
in the same way.

\[ \hat{C} \Psi_1 \gamma^\mu \Psi_2 \hat{C} = - \bar{\Psi}_2 \gamma^\mu \Psi_1. \]  
(S.25)

Now, let \( \Psi_1 = \Psi \) while \( \Psi_2 = i \partial_\mu \Psi \), then

\[ \hat{C} \bar{\Psi}(i \gamma^\mu \partial_\mu) \hat{C} = -i \partial_\mu \bar{\Psi} \gamma^\mu \Psi + i \bar{\Psi} \gamma^\mu \partial_\mu \Psi - i \partial_\mu (\bar{\Psi} \gamma^\mu \Psi) \]  
(S.26)

where the second term is a total derivative. Altogether,

\[ \hat{C} \bar{\Psi}(i \not{\partial} - m) \hat{C} = + \bar{\Psi}(i \not{\partial} - m) \Psi + \text{a total derivative.} \]  
(S.27)

Problem 2(a):
The \( \gamma^0 \) matrix commutes with itself but anticommutes with the space-indexed \( \gamma^{1,2,3} \). At the same time, the parity reflects the space coordinates but not the time coordinate, \( x \to x' = -x \) but \( t \to t' = +t \), hence the new space and time derivatives are related to the old derivatives as \( \nabla' = -\nabla \) but \( \partial_0' = +\partial \). Together, these two facts give us

\[ \not{\partial}' \times \gamma^0 = (\gamma^0 \partial_0' + \vec{\gamma} \cdot \nabla') \gamma^0 = \gamma^0(\gamma^0 \partial_0' - \vec{\gamma} \cdot \nabla') = \gamma^0(+ \gamma^0 \partial_0 + \vec{\gamma} \cdot \nabla) = \gamma^0 \times \not{\partial} \]  
(S.28)

and hence

\[ (i \not{\partial}' - m) \times \gamma^0 = \gamma^0 \times (i \not{\partial} - m). \]  
(S.29)

Combining this formula with eq. (5) for the Dirac field, we find

\[ (i \not{\partial}' - m) \Psi'(x') = (i \not{\partial} - m)( \pm \gamma^0 \Psi(x)) = \pm (i \not{\partial} - m) \gamma^0 \Psi(x) = \pm \gamma^0 (i \not{\partial} - m) \Psi(x) \]  
(S.30)

— the \( (i \not{\partial} - m) \Psi(x) \) transforms under parity precisely like the \( \Psi(x) \) field itself. In other words, the Dirac equation transforms covariantly.
Now consider the Dirac Lagrangian. Taking the Hermitian conjugate of eq. (5) we find
\[
\Psi'(\mathbf{x}, t) = \pm \Psi^\dagger(\mathbf{x}, t)\gamma^0 = \pm \Psi^\dagger(\mathbf{x}, t)\gamma^0
\]  
(S.31)
and hence
\[
\Psi'(\mathbf{x}, t) = \pm \Psi(\mathbf{x}, t)\gamma^0.
\]  
(S.32)
Consequently, the Dirac Lagrangian \( \mathcal{L} = \overline{\Psi}(i \not\partial - m)\Psi \) transforms into
\[
\mathcal{L}(x') = \overline{\Psi}(x') \times (i \not\partial - m)'\Psi'(x') \\
= \pm \overline{\Psi}(x)\gamma^0 \times \pm \gamma^0(i \not\partial - m)\Psi(x) \\
= + \overline{\Psi}(x) \times (i \not\partial - m)\Psi(x) \\
= \mathcal{L}(x).
\]  
(S.33)
In other words, the Dirac Lagrangian is invariant modulo \( x \to x' = (-x, +t) \), and the Dirac action \( S = \int d^4x \mathcal{L} \) is invariant.

**Problem 2(b):**
In the last homework (set#7, problem 3, parts (a–c)) we saw that in the Weyl basis of \( \gamma \) matrices,
\[
u(p, s) = \begin{pmatrix} +\sqrt{E - \mathbf{p} \cdot \mathbf{\sigma}} \xi_s \\ \sqrt{E + \mathbf{p} \cdot \mathbf{\sigma}} \xi_s \end{pmatrix}, \quad v(p, s) = \begin{pmatrix} +\sqrt{E - \mathbf{p} \cdot \mathbf{\sigma}} \eta_s \\ -\sqrt{E + \mathbf{p} \cdot \mathbf{\sigma}} \eta_s \end{pmatrix}.
\]  
(S.34)
The parity symmetry reverses the particle momenta \( p \to -p \) but it does not reverses spins (or any other angular momenta), thus \( s \to s \). Applying such parity reversal to the Dirac spinors (S.34), we find
\[
u(p, s) \to u(-p, +s) = \begin{pmatrix} +\sqrt{E + \mathbf{p} \cdot \mathbf{\sigma}} \xi_s \\ +\sqrt{E - \mathbf{p} \cdot \mathbf{\sigma}} \xi_s \end{pmatrix} \\
= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} +\sqrt{E - \mathbf{p} \cdot \mathbf{\sigma}} \xi_s \\ +\sqrt{E + \mathbf{p} \cdot \mathbf{\sigma}} \xi_s \end{pmatrix} \\
= \gamma^0 \times u(p, s),
\]  
(S.35)
\[
v(p, s) \rightarrow v(-p, +s) = \begin{pmatrix} +\sqrt{E + p \cdot \sigma \xi_s} \\ -\sqrt{E - p \cdot \sigma \xi_s} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} +\sqrt{E - p \cdot \sigma \xi_s} \\ -\sqrt{E + p \cdot \sigma \xi_s} \end{pmatrix} = -\gamma^0 \times v(p, s).
\]
(S.36)

**Problem 2(c):**

Now let’s apply parity to the quantum Dirac field

\[
\hat{\Psi}(x, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left( e^{-itE_p + i\mathbf{x} \cdot \mathbf{p}} \times u(p, s) \times \hat{a}_{p, s} + e^{+itE_p - i\mathbf{x} \cdot \mathbf{p}} \times v(p, s) \times \hat{b}_{p, s}^\dagger \right).
\]

(S.37)

Everything besides the \( \hat{a}_{p, s} \) and \( \hat{b}_{p, s}^\dagger \) operators in this expansion is a c-number, hence the expression on the left hand side of eq. (6) expands to

\[
\hat{P} \hat{\Psi}(x, t) \hat{P} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left( e^{-itE_p + i\mathbf{x} \cdot \mathbf{p}} \times \hat{P} \hat{a}_{p, s} \hat{P} + e^{+itE_p - i\mathbf{x} \cdot \mathbf{p}} \times \hat{P} \hat{b}_{p, s}^\dagger \hat{P} \right).
\]

(S.38)

At the same time, for the right hand side of eq. (6) we have

\[
\pm \gamma^0 \hat{\Psi}(-x, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left( \pm e^{-itE_p - i\mathbf{x} \cdot \mathbf{p}} \times \gamma^0 u(p, s) \times \hat{a}_{p, s} \right.
\]

\[
\left. \pm e^{+itE_p + i\mathbf{x} \cdot \mathbf{p}} \times \gamma^0 v(p, s) \times \hat{b}_{p, s}^\dagger \right)
\]

\(\langle\text{using part (b)}\rangle\)

\[
= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left( \pm e^{-itE_p - i\mathbf{x} \cdot \mathbf{p}} \times u(-p, s) \times \hat{a}_{-p, s} \right.
\]

\[
\left. \pm e^{+itE_p + i\mathbf{x} \cdot \mathbf{p}} \times v(-p, s) \times \hat{b}_{-p, s}^\dagger \right)
\]

(S.39)

\(\langle\text{changing } \int \text{ variable } \mathbf{p} \rightarrow -\mathbf{p}\rangle\)

By eq. (6), the right hand sides of eqs. (S.38) and (S.39) must be equal to each other. Since the Dirac plane waves \( e^{-ipx} u(p, s) \) and \( e^{+ipx} v(p, s) \) are linearly independent from each other,
this means
\[ \hat{\mathbf{P}} \hat{a}_{p,s} \hat{\mathbf{P}} = \pm \hat{a}_{-p,s} \quad \text{and} \quad \hat{\mathbf{P}} \hat{b}_{p,s} \hat{\mathbf{P}} = \mp \hat{b}_{-p,s}. \] (7a)

The rest of eq. (7) follows by hermitian conjugation: Since \( \hat{\mathbf{P}}\dagger = \hat{\mathbf{P}}^{-1} = \hat{\mathbf{P}}, \)
\[ \hat{\mathbf{P}} \hat{a}_{p,s} \hat{\mathbf{P}} = \left( \hat{\mathbf{P}} \hat{a}_{p,s} \hat{\mathbf{P}} \right)\dagger = \pm \hat{a}_{-p,s}, \] (7b)
\[ \hat{\mathbf{P}} \hat{b}_{p,s} \hat{\mathbf{P}} = \left( \hat{\mathbf{P}} \hat{b}_{p,s} \hat{\mathbf{P}} \right)\dagger = \mp \hat{b}_{-p,s}. \]

Finally, eqs. (8) follow from eqs. (7) and parity-invariance of the vacuum state, \( \hat{\mathbf{P}} |0\rangle = |0\rangle. \)
Indeed,
\[ \hat{\mathbf{P}} |F(p, s)\rangle = \hat{\mathbf{P}} \times \hat{a}_{p,s} |0\rangle = \hat{\mathbf{P}} \hat{a}_{p,s} \hat{\mathbf{P}} \times |0\rangle = \pm |F(-p, +s)\rangle, \] (S.40)
\[ \hat{\mathbf{P}} |\bar{F}(p, s)\rangle = \hat{\mathbf{P}} \times \hat{b}_{p,s} |0\rangle = \hat{\mathbf{P}} \hat{b}_{p,s} \hat{\mathbf{P}} \times |0\rangle = \mp |\bar{F}(-p, +s)\rangle. \] (S.41)

Problem 3(a):
Consider a state \( \hat{a}\dagger (+p_{\text{red}}, s_1) \hat{b}\dagger (-p_{\text{red}}, s_2) |0\rangle \) of one fermion and one antifermion with definite reduced momentum and spins. The charge conjugation operator \( \hat{\mathbf{C}} \) turns this state into
\[ \hat{\mathbf{C}} \times \hat{a}\dagger (+p_{\text{red}}, s_1) \hat{b}\dagger (-p_{\text{red}}, s_2) |0\rangle \]
\[ = \hat{\mathbf{C}} \hat{a}\dagger (+p_{\text{red}}, s_1) \hat{\mathbf{C}} \times \hat{b}\dagger (-p_{\text{red}}, s_2) \hat{\mathbf{C}} \times |0\rangle \]
\[ = \hat{b}\dagger (+p_{\text{red}}, s_1) \hat{a}\dagger (-p_{\text{red}}, s_2) |0\rangle \]
\[ = -\hat{a}\dagger (-p_{\text{red}}, s_2) \hat{b}\dagger (+p_{\text{red}}, s_1) |0\rangle. \] (S.42)

Let’s plug this formula into eq. (10):
\[ \hat{\mathbf{C}} \times |B(p_{\text{tot}})\rangle = \int \frac{d^3p_{\text{red}}}{(2\pi)^3} \sum_{s_1, s_2} \psi(p_{\text{red}}, s_1, s_2) \times \hat{\mathbf{C}} \hat{a}\dagger (+p_{\text{red}}, s_1) \hat{b}\dagger (-p_{\text{red}}, s_2) |0\rangle \]
\[ = \int \frac{d^3p_{\text{red}}}{(2\pi)^3} \sum_{s_1, s_2} \psi(p_{\text{red}}, s_1, s_2) \times -\hat{a}\dagger (-p_{\text{red}}, s_2) \hat{b}\dagger (+p_{\text{red}}, s_1) |0\rangle \] (S.43)
\[ \langle | \text{change variables } p_{\text{red}} \to -p_{\text{red}} \text{ and } s_1 \leftrightarrow s_2 \rangle \]
\[ = \int \frac{d^3p_{\text{red}}}{(2\pi)^3} \sum_{s_1, s_2} -\psi(-p_{\text{red}}, s_2, s_1) \times \hat{a}\dagger (+p_{\text{red}}, s_1) \hat{b}\dagger (-p_{\text{red}}, s_2) |0\rangle. \]
In terms of the bound state’s wave function $\psi$, this action of the C-parity operator $\hat{C}$ is equivalent to

$$\hat{C} \psi(p_{\text{red}}, s_1, s_2) = -\psi(-p_{\text{red}}, s_2, s_1).$$  \hfill (S.44)

For a bound state with a definite orbital angular momentum $L$,

$$\psi(-p_{\text{red}}, s_1, s_2) = (-1)^L \times \psi(+p_{\text{red}}, s_1, s_2).$$  \hfill (S.45)

Likewise, for a bound state with a definite net spin $S$,

$$\psi(p_{\text{red}}, s_2, s_1) = (-1)^{1-S} \psi(p_{\text{red}}, s_1, s_2).$$  \hfill (S.46)

Plugging these two formulae into eq. (S.44) for the C-parity, we obtain

$$\hat{C} \psi(p_{\text{red}}, s_1, s_2) = -\psi(-p_{\text{red}}, s_2, s_1)$$

$$= -(-1)^L \psi(+p_{\text{red}}, s_2, s_1)$$

$$= -(-1)^L(-1)^{1-S} \psi(+p_{\text{red}}, s_1, s_2).$$  \hfill (S.47)

In other words, the bound state has definite C-parity

$$C = -(-1)^L(-1)^{1-S} = (-1)^L \times (-1)^S,$$  \hfill (S.48)

Q.E.D.

Problem 3(b):

Now consider how the P-parity (reflection of space) acts on the one-fermion+one-antifermions state $\hat{a}^\dagger(+p_{\text{red}}, s_1)\hat{b}^\dagger(-p_{\text{red}}, s_2) |0\rangle$:

$$\hat{P} \times \hat{a}^\dagger(+p_{\text{red}}, s_1)\hat{b}^\dagger(-p_{\text{red}}, s_2) |0\rangle = \hat{P} \hat{a}^\dagger(+p_{\text{red}}, s_1)\hat{P} \times \hat{b}^\dagger(-p_{\text{red}}, s_2)\hat{P} \times |0\rangle$$

$$= (\pm 1)\hat{a}^\dagger(-p_{\text{red}}, s_1) \times (\mp)\hat{b}^\dagger(+p_{\text{red}}, s_2) |0\rangle$$

$$= -\hat{a}^\dagger(-p_{\text{red}}, s_1)\hat{b}^\dagger(+p_{\text{red}}, s_2) |0\rangle.$$

where the overall $-\text{sign}$ comes from opposite intrinsic parities of the fermion and the antifermion. Again, we plug this formula into eq. (10) and then change the integration variable.
\( \mathbf{p}_{\text{red}} \rightarrow -\mathbf{p}_{\text{red}} \) — but this time we do not swap the spins \( s_1 \) and \( s_2 \):

\[
\hat{P} \times |B(p_{\text{tot}})\rangle = \int \frac{d^3 p_{\text{red}}}{(2\pi)^3} \sum_{s_1,s_2} \psi(\mathbf{p}_{\text{red}}, s_1, s_2) \times \hat{P} \hat{a}^\dagger (+\mathbf{p}_{\text{red}}, s_1) \hat{b}^\dagger (-\mathbf{p}_{\text{red}}, s_2) |0\rangle
\]

\[
= \int \frac{d^3 p_{\text{red}}}{(2\pi)^3} \sum_{s_1,s_2} \psi(\mathbf{p}_{\text{red}}, s_1, s_2) \times -\hat{a}^\dagger (-\mathbf{p}_{\text{red}}, s_1) \hat{b}^\dagger (+\mathbf{p}_{\text{red}}, s_2) |0\rangle \tag{S.50}
\]

\[
= \int \frac{d^3 p_{\text{red}}}{(2\pi)^3} \sum_{s_1,s_2} -\psi(-\mathbf{p}_{\text{red}}, s_1, s_2) \times \hat{a}^\dagger (+\mathbf{p}_{\text{red}}, s_1) \hat{b}^\dagger (-\mathbf{p}_{\text{red}}, s_2) |0\rangle,
\]

In terms of the wave-function \( \psi \), this action of the P-parity operator means

\[
\hat{P} \psi(\mathbf{p}_{\text{red}}, s_1, s_2) = -\psi(-\mathbf{p}_{\text{red}}, s_1, s_2). \tag{S.51}
\]

For a bound state with a definite angular momentum, this gives us

\[
\hat{P} \psi(\mathbf{p}_{\text{red}}, s_1, s_2) = -\psi(-\mathbf{p}_{\text{red}}, s_1, s_2) = -( -1)^L \psi(\mathbf{p}_{\text{red}}, s_1, s_2) \tag{S.52}
\]

and hence definite P-parity

\[
P = -(-1)^L, \tag{S.53}
\]

Q.E.D.

Problem 3(c):
Finally, consider the positronium atom decaying into photons. Since the EM interactions are symmetric under charge conjugations, the EM processes such as \( e^- + e^+ \rightarrow \text{photons} \) conserve C-parity. A photon of any momentum or polarization has \( C = -1 \), so the net C-parity of an \( n \)-photon finite state is \((-1)^n\). Consequently, if the initial electron and positron are in a bound state with \( C = +1 \) they must annihilate into an even number of photons, \( e^- + e^+ \rightarrow 2\gamma, 4\gamma, 6\gamma, \ldots \). But if the bound state has \( C = -1 \), the electron and the positron must annihilate into an odd number of photons, \( e^- + e^+ \rightarrow 3\gamma, 5\gamma, \ldots \) (Annihilation into a single photon is forbidden because of \( p_{\text{net}}^2 < E_{\text{net}}^2 \).)

11
The ground state of a hydrogen-like positronium ‘atom’ is 1S, meaning $n_{\text{rad}} = 1$ and $L = 0$. Due to spins, there are actually 4 almost-degenerate 1S states; the hyperfine structure splits them into a 1S$_3$ triplet and a 1S$_1$ singlet of the net spin. According to eq. (S.48), the triplet states have $C = (-1)^L (-1)^S = (-1)^0 (-1)^1 = -1$ while the singlet state has $C = (-1)^L (-1)^S = (-1)^0 (-1)^0 = +1$. Consequently, the singlet $S = 0$ state decays into an even number of photons,

$$
(e^- + e^+)_{1S_1} \rightarrow 2\gamma, 4\gamma, \ldots,
$$

while the triplet $S = 1$ states decay into odd numbers of photons,

$$
(e^- + e^+)_{1S_3} \rightarrow 3\gamma, 5\gamma, \ldots.
$$

This difference affects the net decay rate of each state because QED (Quantum Electrodynamics) has a rather small coupling constant $\alpha = (e^2/4\pi) \approx 1/137$. For each photon in the final state, the decay amplitude carries a factor of $e$, so the decay rate of a positronium atom into $n$ photons $\Gamma(e^-e^+ \rightarrow n\gamma)$ is $O(\alpha^n)$. Consequently, the $S = 0$ positronium state usually decays into just 2 photons while decays into 4, 6, or more photons are allowed but much less common. Likewise, the $S = 1$ positronium states usually decays into 3 photons while decays into 5 or more photons are allowed but rare. More over, the decay rate into 3 photons is much slower than the decay rate into just 2 photons,

$$
\frac{\Gamma((e^- + e^+)_{1S_3} \rightarrow 3\gamma)}{\Gamma((e^- + e^+)_{1S_1} \rightarrow 2\gamma)} = \frac{O(\alpha^3)}{O(\alpha^2)} = O(\alpha),
$$

hence the net decay rate of an $S = 1$ state into anything it can decay to — i.e., into any odd number of photons — is much slower then the net decay rate of the $S = 0$ state,

$$
\frac{\Gamma((e^- + e^+)_{1S_3} \rightarrow \text{anything})}{\Gamma((e^- + e^+)_{1S_1} \rightarrow \text{anything})} = O(\alpha) \ll 1.
$$

And that’s why the $S = 1$ states have much longer lifetimes than the $S = 0$ state.
Problem 4:

Let’s start with the parity $P$:

$$\hat{P}\hat{\Psi}(x,t)\hat{P} = \pm \gamma^0 \hat{\Psi}(-x,t) = \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} \hat{\psi}_L(-x,t) \\ \hat{\psi}_R(-x,t) \end{pmatrix} = \begin{pmatrix} \pm \hat{\psi}_R(-x,t) \\ \pm \hat{\psi}_L(-x,t) \end{pmatrix}, \quad (S.58)$$

and therefore

$$\hat{P}\hat{\psi}_L(x,t)\hat{P} = \pm \hat{\psi}_R(-x,+t), \quad \hat{P}\hat{\psi}_R(x,t)\hat{P} = \pm \hat{\psi}_L(-x,+t). \quad (S.59)$$

Note that the parity exchanges the left-handed and the right-handed Weyl spinors. Consequently, in parity-symmetric theories the two kinds of Weyl spinor fields must pair up to form Dirac spinors. (But in theories without the P symmetry, the LH and the RH Weyl spinors fields may be independent from each other.)

Now consider the charge conjugation $C$:

$$\hat{C}\hat{\Psi}(x)\hat{C} = \gamma^2 \hat{\Psi}^*(x) = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \times \begin{pmatrix} \hat{\psi}_L^*(x) \\ \hat{\psi}_R^*(x) \end{pmatrix} = \begin{pmatrix} +\sigma_2 \hat{\psi}_R^*(x) \\ -\sigma_2 \hat{\psi}_L^*(x) \end{pmatrix}, \quad (S.60)$$

and therefore

$$\hat{C}\hat{\psi}_L(x)\hat{C} = +\sigma_2 \hat{\psi}_R^*(x), \quad \hat{C}\hat{\psi}_R(x)\hat{C} = -\sigma_2 \hat{\psi}_L^*(x). \quad (S.61)$$

Similar to parity, the charge conjugation exchanges the LH and the RH Weyl spinors with each other, so C-symmetric theories must have Dirac spinor fields rather than independent LH and RH Weyl spinor fields.

Finally, consider the combined CP symmetry. Combining eqs. (S.59) and (S.61), we obtain

$$\hat{C}\hat{P}\hat{\psi}_L(x,t)\hat{P}\hat{C} = \pm \hat{C}\hat{\psi}_R(-x,+t)\hat{C} = \pm \sigma_2 \hat{\psi}_L^*(-x,+t), \quad (S.62)$$

$$\hat{C}\hat{P}\hat{\psi}_R(x,t)\hat{P}\hat{C} = \pm \hat{C}\hat{\psi}_L(-x,+t)\hat{C} = \mp \sigma_2 \hat{\psi}_R^*(-x,+t). \quad (S.63)$$

We see that unlike the separate C and P symmetries, the combined CP does not mix the LH and the RH Weyl spinors with each other. Consequently, CP-symmetric theories (without separate C and P symmetries) may have chiral fermions, that is, independent LH and RH Weyl spinor fields which do not pair up into Dirac spinors. In particular, the LH Weyl fields and the RH Weyl fields may have different charges or other quantum numbers.
Problem 5(a):
Despite anticommutativity of the fermionic fields, the Hermitian conjugation of an operator product reverses the order of operators without any extra sign factors, thus \((\Psi_\alpha^\dagger \Psi_\beta)^\dagger = +\Psi_\beta^\dagger \Psi_\alpha\). Consequently, for any \(4 \times 4\) matrix \(\Gamma\), \((\Psi^\dagger \Gamma \Psi)^\dagger = +\Psi^\dagger \Gamma^\dagger \Psi\), and hence \((\overline{\Psi} \Gamma \Psi)^\dagger = \overline{\Psi}^\dagger \Gamma^\dagger \Psi\) where \(\Gamma = \gamma^0 \Gamma^\dagger \gamma^0\) is the Dirac conjugate of \(\Gamma\).

Now consider the 16 matrices which appear in the bilinears (12). Obviously \(\overline{1} = +1\) and this gives us \(S^\dagger = +S\). We saw in class that \(\gamma^\mu = +\gamma^\mu\), and this gives us \((V^\mu)^\dagger = +V^\mu\). We also saw that \(\frac{i}{2} \gamma^{\mu \nu} = -\frac{i}{2} \gamma^{\nu \mu}\) and \(= +\frac{i}{2} \gamma^{\mu \nu}\), and this gives us \((T^{\mu \nu})^\dagger = +T^{\mu \nu}\). As to the \(\gamma^5\) matrix, we saw in the last homework (problem 2) that it’s Hermitian and anticommutes with all the \(\gamma^\mu\). Hence \(\overline{\gamma^5} = \gamma^0 (\gamma^5)^\dagger \gamma^0 = +\gamma^0 \gamma^5 \gamma^0 = -\gamma^5 \implies i \overline{\gamma^5} = +i \gamma^5\), which gives us \(P^\dagger = +P\). Finally, \(\overline{\gamma^5 \gamma^\mu} = \gamma^5 \gamma^\mu = +\gamma^5 \gamma^\mu\), which gives us \((A^\mu)^\dagger = +A^\mu\). Thus, by inspection, all the bilinears (12) are Hermitian. \(\textit{Q.E.D.}\)

Problem 5(b):
Under a continuous Lorentz symmetry \(x \mapsto x' = Lx\), the Dirac spinor field and its conjugate transform according to

\[ \Psi'(x') = M(L) \Psi(x = L^{-1}x'), \quad \overline{\Psi}'(x') = \overline{\Psi}(x = L^{-1}x') M^{-1}(L), \]  

(S.64)

hence any bilinear \(\overline{\Psi} \Gamma \Psi\) transforms according to

\[ \overline{\Psi}'(x') \Gamma \Psi(x') = \overline{\Psi}(x) \Gamma' \Psi(x) \]  

(S.65)

where

\[ \Gamma' = M^{-1}(L) \Gamma M(L). \]  

(S.66)

Obviously for \(\Gamma = 1\), \(\Gamma' = M^{-1}M = 1\), which makes \(S\) a Lorentz scalar.

For \(\Gamma = \gamma^\mu\), we saw in class that \(\Gamma' = M^{-1} \gamma^\mu M = L^\mu_\nu \gamma^\nu\) — see my notes on Dirac spinors, eq. (21). Consequently \(V'^\mu = L^\mu_\nu V^\nu\), which makes \(V^\mu\) a Lorentz vector.

For \(\Gamma = \gamma^\mu \gamma^\nu\), \(M^{-1} \gamma^\mu \gamma^\nu M = (M^{-1} \gamma^\mu M)(M^{-1} \gamma^\nu M) = L^\mu_\kappa \gamma^\kappa \times L^\nu_\lambda \gamma^\lambda\). Similar transformation works for \(\Gamma = \frac{i}{2} \gamma^{\mu \nu}\): \(\Gamma' = L^\mu_\kappa L^\nu_\lambda \times \frac{i}{2} \gamma^{\kappa \lambda}\). This makes \(T^{\mu \nu}\) a Lorentz tensor (with two antisymmetric indices).
Finally, the $\gamma^5$ commutes with even products of the $\gamma^\mu$ matrices and hence with $M(L) = \exp\left(\frac{i}{4} \Theta_{\mu\nu} \gamma^\mu \gamma^\nu \right)$. Consequently, $M^{-1} \gamma^5 M = \gamma^5$, which makes $P$ a Lorentz scalar. Likewise, $M^{-1}(\gamma^\mu \gamma^5)M = (M^{-1} \gamma^\mu M) \gamma^5 = L^\mu \gamma^\nu \gamma^5$, which makes $A^\mu$ a Lorentz vector. Q.E.D.

Problem 5(c):
I problem (2) we saw that the Dirac fields transform under parity as
\[
\Psi(x') = \pm \gamma^0 \Psi(x), \quad \overline{\Psi}'(x') = \pm \overline{\Psi}(x) \gamma^0. \tag{S.67}
\]
Consequently, the Dirac bilinears transform as
\[
P : \overline{\Psi} \Gamma \Psi \bigg|_x \mapsto \overline{\Psi}' \Gamma \Psi' \bigg|_{x'} = \overline{\Psi}(\gamma^0 \Gamma \gamma^0) \Psi \bigg|_x. \tag{S.68}
\]
By inspection, out of 16 possible $\Gamma$ matrices, 1, $\gamma^0$, $\gamma[i\gamma^j]$, and $\gamma^5\gamma^i$ commute with the $\gamma^0$, while $\gamma^i$, $\gamma^0\gamma^i$, $\gamma^5\gamma^0$, and $\gamma^5$ anticommute with the $\gamma^0$. Therefore,
- the $S$, $V^0$, $T^{ij}$, and $A^i$ remain invariant under parity, while
- the $V^i$, $T^{0i}$, $A^0$, and $P$ change their signs.

From the 3D point of view, this means that $S$ and $V^0$ are true scalars, $P$ and $A^0$ are pseudo-scalars, $V$ is a true or polar vector, $A$ is a pseudo-vector or axial vector, and the tensor $T$ contains one true vector $T^{0i}$ and one axial vector $\frac{1}{2} \epsilon^{ijk} T^{jk}$. In space-time terms, we call $S$ a true (Lorentz) scalar, $P$ a (Lorentz) pseudoscalar, $V^\mu$ a true (Lorentz) vector, and $A^\mu$ an axial (Lorentz) vector. Pedantically speaking, $T^{\mu\nu}$ is a true Lorentz tensor while $\tilde{T}^{\kappa\lambda} = \frac{1}{2} \epsilon^{\kappa\lambda\mu\nu} T_{\mu\nu}$ is a Lorentz pseudo-tensor, but few people bother with this distinction.

Problem 5(d):
In problem 1 we saw that the charge conjugation symmetry acts on Dirac fields as
\[
\Psi(x) \rightarrow \Psi'(x) = \gamma^2 \Psi^*(x) = \gamma^2 (\Psi^\dagger(x))^\top,
\quad \overline{\Psi}(x) \rightarrow \overline{\Psi}'(x) = \overline{\Psi}^*(x) \gamma^2 = \overline{\Psi}^\top(x) \gamma^0 \gamma^2 = -\overline{\Psi}^\top(x) \gamma^2 \gamma^0. \tag{S.69}
\]
Consequently, for any Dirac bilinear $\overline{\Psi} \Gamma \Psi$,
\[
\overline{\Psi}' \Gamma \Psi' = -\overline{\Psi}^\top \gamma^2 \gamma^0 \Gamma \gamma^2 (\Psi^\dagger) = +\Psi^\dagger (\gamma^2 \gamma^0 \Gamma \gamma^2 \overline{\Psi}^\top = +\overline{\Psi} \gamma^0 \Gamma \gamma^2 \gamma^2 \Psi = \overline{\Psi} \Gamma \gamma^0 \gamma^2. \tag{S.70}
\]

The second equality here follows by transposition of the Dirac “sandwich” $\Psi^\top \cdots (\Psi^\dagger)^\top$, which carries an extra minus sign because the fermionic fields $\Psi$ and $\Psi^*$ anticommute with
each other (in the classical limit). The third equality follows from \((\gamma^0)\top = +\gamma^0\), \((\gamma^2)\top = +\gamma^2\), and \(\Psi\dagger = \Psi\gamma^0\).

**Problem 5(e):**

By inspection, \(1^c \equiv \gamma^0\gamma^2\gamma^0\gamma^2 = +1\). The \(\gamma_5\) matrix is symmetric and commutes with the \(\gamma^0\gamma^2\), hence \(\gamma^c_5 = +\gamma_5\). Among the four \(\gamma_\mu\) matrices, the \(\gamma_1\) and \(\gamma_3\) are anti-symmetric and commute with the \(\gamma^0\gamma^2\) while the \(\gamma_0\) and \(\gamma_2\) are symmetric but anti-commute with the \(\gamma^0\gamma^2\); hence, for all four \(\gamma_\mu\), \(\gamma^c_\mu = -\gamma_\mu\). Finally, because of the transposition involved, \((\gamma_\mu\gamma_\nu)^c = \gamma^c_\nu\gamma^c_\mu = +\gamma_\nu\gamma_\mu\), hence \((\frac{i}{2}\gamma[\mu\gamma_\nu])^c = +\frac{i}{2}\gamma[\nu\gamma_\mu] = -\frac{i}{2}\gamma[\mu\gamma_\nu]\). Likewise, \((\gamma^5\gamma_\mu)^c = (\gamma^5)^c = -\gamma^5\gamma^5 = +\gamma^5\gamma_\mu\).

Therefore, according to eq. (S.70), the scalar \(S\), the pseudoscalar \(P\), and the axial vector \(A_\mu\) are C–even, while the vector \(V_\mu\) and the tensor \(T_{\mu\nu}\) are C–odd.