Problem 2(a):
There are two tree diagrams for the $e^- e^+ \rightarrow S \gamma$ process, namely

These two diagrams are related by $t \leftrightarrow u$ crossing, and also by the charge conjugation (which exchanges the initial $e^-$ and $e^+$). The net tree-level amplitude is

\[
\mathcal{M}_{\text{tree}} = \mathcal{E}_{k,\lambda}^\mu(\gamma) \times \mathcal{M}_\mu, \\
\mathcal{M}_\mu = \mathcal{M}_1^\mu + \mathcal{M}_2^\mu, \\
\mathcal{M}_1^\mu = -i \bar{v}(e^+) (-ig) \frac{i}{\slashed{q} - m_e} (ie\gamma^\mu) u(e^-) \\
= \frac{eg}{t - m^2} \times \bar{v}(\slashed{q} + m_e) \gamma^\mu u, \\
\mathcal{M}_2^\mu = -i \bar{v}(e^+) (ie\gamma^\mu) \frac{i}{\slashed{q} - m_e} (-ig) u(e^-) \\
= \frac{eg}{u - m^2} \times \bar{v}\gamma^\mu(\slashed{q} + m_e) u,
\]

where

\[
q = p_+ - k_\gamma = k_s - p_+ \quad \text{and} \quad \bar{q} = p_+ - k_s = k_\gamma - p_+ .
\]
Problem 2(b):
The Ward identity for the one-photon amplitude (S.2) says \( k_\gamma^\mu \times M_\mu = 0 \). To verify it, let’s start with the first diagram:

\[
k_\gamma^\mu \times \bar{v}(q + m_e)_{\gamma\mu}u = \bar{v}(q + m_e) k_\gamma^\mu u \\
= \bar{v}(p_- - k_\gamma + m_e) k_\gamma^\mu u \\
= \bar{v}(p_- + m_e) k_\gamma^\mu u \quad \langle \text{because } k_\gamma k_\gamma = k_\gamma^2 = 0 \rangle \\
= \bar{v} \left( 2(p_- k_\gamma) - k_\gamma (p_- - m_e) \right) u \\
= 2(p_- k_\gamma) \times \bar{v} u - 0 \quad \langle \text{because } (p_- - m_e) \times u(e^-) = 0 \rangle \\
= (m_e^2 - t) \times \bar{v} u ,
\]

and hence

\[
k_\gamma^\mu \times M_{1\mu} = -e g \times \bar{v} u . \tag{S.5}
\]

We see that by itself, the first diagram does not satisfy the Ward entity. Instead, we need to add the second diagram’s contribution

\[
k_\gamma^\mu \times \bar{v}\gamma\mu(q + m_e)u = \bar{v} k_\gamma^\mu (q + m_e)u \\
= \bar{v} k_\gamma^\mu (k_\gamma - p_+ + m_e)u \\
= \bar{v} k_\gamma^\mu (-p_+ + m_e)u \quad \langle \text{because } k_\gamma k_\gamma = k_\gamma^2 = 0 \rangle \\
= \bar{v} \left( -2(p_+ k_\gamma) + k_\gamma (p_+ + m_e) \right) u \\
= -2(p_+ k_\gamma) \times \bar{v} u + 0 \quad \langle \text{because } \bar{v} e^+ \times (p_+ + m_e) = 0 \rangle \\
= (u - m_e^2) \times \bar{v} u ,
\]

and hence

\[
k_\gamma^\mu \times M_{2\mu} = +e g \times \bar{v} u . \tag{S.6}
\]

Again, the second diagram does not satisfy the Ward identity by itself, but the net amplitude does:

\[
k_\gamma^\mu \times (M_\mu = M_{1\mu} + M_{2\mu}) = 0 . \tag{S.8}
\]
Problem 2(c):
Thanks to the Ward identity, summing $|\mathcal{M}|^2$ over the photon’s polarizations is easy:

$$
\sum_{\lambda} |\mathcal{M}|^2 = -\mathcal{M}^\mu \mathcal{M}_\mu^* \quad \langle \langle \text{see my notes on Ward identities} \rangle \rangle 
$$

$$
= -\mathcal{M}_1^\mu \mathcal{M}_{1\mu}^* - \mathcal{M}_2^\mu \mathcal{M}_{2\mu}^* - 2 \text{Re} \left( \mathcal{M}_1^\mu \mathcal{M}_2^\mu \right) 
$$

$$
= -\frac{e^2 g^2}{(t - m_e^2)^2} \times \bar{v}(\not{q} + m_e) \gamma^\mu u \times \bar{u}(\not{q} + m_e) v 
$$

$$
- \frac{e^2 g^2}{(u - m_e^2)^2} \times \bar{v}\gamma^\mu (\not{q} + m_e) u \times \bar{u}(\not{q} + m_e) \gamma_\mu u 
$$

$$
- \frac{2 e^2 g^2}{(t - m_e^2)(u - m_e^2)} \times \text{Re} \left( \bar{v}(\not{q} + m_e) \gamma^\mu u \times \bar{u}(\not{q} + m_e) \gamma_\mu v \right). 
$$

(S.9)

And given this formula, averaging over the electron’s and positron’s spins produces Dirac traces according to

$$
|\mathcal{M}|^2 \equiv \frac{1}{4} \sum_{s_-,s_+} \sum_{\lambda} |\mathcal{M}|^2 = e^2 g^2 \left( \frac{A_{11}}{t - m_e^2} + \frac{A_{22}}{u - m_e^2} + \frac{2 \text{Re} A_{12}}{(t - m_e^2)(u - m_e^2)} \right) \quad \text{(S.10)}
$$

where

$$
A_{11} = -\frac{1}{4} \text{Tr} \left( (p_+ - m_e)(\not{q} + m_e) \gamma^\mu (\not{p}_- - m_e) \gamma_\mu (\not{q} + m_e) \right), 
$$

$$
A_{22} = -\frac{1}{4} \text{Tr} \left( (p_+ - m_e) \gamma^\mu (\not{q} + m_e)(\not{p}_- + m_e)(\not{q} + m_e) \gamma_\mu \right), 
$$

$$
A_{12} = -\frac{1}{4} \text{Tr} \left( (p_+ - m_e)(\not{q} + m_e) \gamma^\mu (\not{p}_- + m_e)(\not{q} + m_e) \gamma_\mu \right). \quad \text{(S.11)}
$$

Evaluating these traces is straightforward but tedious. Fortunately, it becomes much simpler when we neglect the electron’s mass. In that limit, the first trace becomes

$$
A_{11} \approx -\frac{1}{4} \text{Tr} (\not{p}_+ \not{q} \gamma^\mu \not{p}_- \gamma_\mu \not{q}) 
$$

$$
= +\frac{1}{2} \text{Tr} (\not{p}_+ \not{q} \not{p}_- \not{q}) \quad \langle \langle \text{using } \gamma^\mu \not{p}_- \gamma_\mu = -2 \not{p}_- \rangle \rangle 
$$

$$
= 4(p_+q)(p_-q) - 2(p_+p_-)q^2 
$$

$$
= (M_s^2 - t) \times t - s \times t = (M_s^2 - t - s) \times t 
$$

$$
\approx u \times t , \quad \text{(S.12)}
$$
where the last two lines follow from

\[ q^2 = t, \]
\[ p_+ p_- = \frac{1}{2} (p_- + p_+)^2 - m_e^2 \approx \frac{s}{2}, \]
\[ p_- q = p_-(p_- - k\gamma) = \frac{1}{2} (p_- - k\gamma)^2 + m_e^2 \approx \frac{t}{2}, \]
\[ p_+ q = p_+(k_s - p_+) = -\frac{1}{2} (p_- - k_s)^2 + \frac{1}{2} M_s^2 - M_s^2 \approx \frac{M_s^2 - t}{2}, \]
\[ s + t + u = M_s^2 + 2m_e^2 \approx M_s^2. \]

Likewise, the second trace becomes

\[ A_{22} \approx -\frac{1}{4} \text{Tr}(\bar{\gamma}_+ \gamma^\mu \tilde{g}_- \bar{\gamma}_\mu) \]
\[ = -\frac{1}{4} \text{Tr}(\gamma_\mu \bar{\gamma}_+ \gamma^\mu \bar{\gamma}_- \bar{\gamma}) \]
\[ = +\frac{1}{2} \text{Tr}(\bar{\gamma}_+ \tilde{g}_- \bar{\gamma}_- \bar{\gamma}) \quad \langle \text{using } \gamma_\mu \bar{\gamma}_+ \gamma^\mu = -2 \bar{\gamma}_+ \rangle \]
\[ = 4(p_+ \tilde{q})(p_- \tilde{q}) - 2(p_+ p_-) \tilde{q}^2 \]
\[ \approx (M_s^2 - u) \times u - s \times u = (M_s^2 - u - s) \times u \]
\[ \approx t \times u, \]

where the last two lines follow from (S.13) and

\[ q^2 = u, \]
\[ p_+ \tilde{q} = p_+(k_\gamma - p_+) = -\frac{1}{2} (k_\gamma - p_+)^2 - m_e^2 \approx -\frac{u}{2}, \]
\[ p_- \tilde{q} = p_-(p_- - k_s) = \frac{1}{2} (p_- - k_s)^2 - \frac{1}{2} M_s^2 + \frac{1}{2} M_s^2 \approx \frac{u - M_s^2}{2}. \]

Finally, the third trace becomes

\[ A_{22} \approx -\frac{1}{4} \text{Tr}(\bar{\gamma}_+ \gamma^\mu \bar{\gamma}_- \gamma_\mu) \]
\[ = -(p_- \tilde{q}) \times \text{Tr}(\bar{\gamma}_+ \tilde{g}) \langle \text{using } \gamma^\mu \bar{\gamma}_- \gamma_\mu = +4(p_- \tilde{q}) \rangle \]
\[ = -4(p_- \tilde{q})(p_+ q) \]
\[ \approx +4(u - M_s^2)(t - M_s^2). \]

Now let’s plug all these traces back into eq. (S.10). Neglecting \( m_e^2 \) in the denominators, we
have
\[
|M|^2 = e^2 g^2 \left( \frac{tu}{t^2} + \frac{ut}{u^2} + \frac{2(t - M_s^2)(u - M_s^2)}{tu} \right)
\]
\[
= e^2 g^2 \times \left( u^2 + t^2 + 2(t - M_s^2)(u - M_s^2) \right)
\]
\[
= e^2 g^2 \times \left( (t + u - M_s^2)^2 + M_s^4 \right)
\]
\[
= e^2 g^2 \times \frac{s^2 + M_s^4}{tu}.
\] (S.17)

Problem 2(d):
Given eq. (S.17) for the spin-averaged and polarization-summed $|M|^2$, calculating the partial cross-section is just the matter of kinematics. In the center of mass frame, the initial electron and positron have $p_+^\mu = (E_e, \pm \mathbf{p})$ where $E_e \approx |\mathbf{p}|$. As to the final photon and scalar, they have equal and opposite 3-momenta but different energies: $k_\gamma^\mu = (\omega, +\mathbf{k})$ while $k_S^\mu = (E_s, -\mathbf{k})$, where $\omega = |\mathbf{k}| \neq E_s = \sqrt{k^2 + M_s^2}$. By energy conservation
\[
\omega + E_s = 2E_e = \sqrt{s}.
\] (S.18)

To solve this equation, we rewrite it as
\[
\omega^2 + M_s^2 = E_s^2 = (\sqrt{s} - \omega)^2 = s - 2\sqrt{s} \times \omega + \omega^2,
\] (S.19)

which gives us
\[
\omega = \frac{s - M_s^2}{2\sqrt{s}} \implies E_s = \frac{s + M_s^2}{2\sqrt{s}}.
\] (S.20)

Given all these momenta,
\[
t = -2(p_- k_\gamma) = -2E_e \omega + 2\mathbf{p} \cdot \mathbf{k} \approx -2E_e \omega \times (1 - \cos \theta) = -\frac{1}{2}(s - M_s^2) \times (1 - \cos \theta),
\]
\[
u = -2(p_+ k_\gamma) = -2E_e \omega - 2\mathbf{p} \cdot \mathbf{k} \approx -2E_e \omega \times (1 + \cos \theta) = -\frac{1}{2}(s - M_s^2) \times (1 + \cos \theta),
\] (S.21)
and plugging these formulae into eq. (S.17) gives us

$$|\mathcal{M}|^2 = 4e^2g^2 \times \frac{s^2 + M_s^4}{(s - M_s^2)^2} \times \frac{1}{\sin^2 \theta}.$$  \hfill (S.22)

In the center-of-mass frame, the partial cross-section of $2$ particles $\rightarrow 2$ particles scattering is given by

$$\frac{d\sigma}{d\Omega_{\text{cm}}} = \frac{|\mathcal{M}|^2}{64\pi^2 s} \times \frac{|p'|}{|p|}. \hfill (S.23)$$

For the problem at hand, the inelasticity factor $|p'|/|p|$ is

$$\frac{|k|}{|p|} \approx \frac{\omega}{E_e} = \frac{s - M_s^2}{s}. \hfill (S.24)$$

Combining this factor with eq. (S.22), we finally arrive at the following formula for the partial cross-section:

$$\frac{d\sigma(e^-e^+\rightarrow \gamma S)}{d\Omega_{\text{c.m.}}} = \frac{\alpha g^2}{4\pi} \times \frac{s^2 + M_s^4}{s^2(s - M_s^2)} \times \frac{1}{\sin^2 \theta}.$$  \hfill (S.25)

Note the forward-backward symmetry $\theta \leftrightarrow \pi - \theta$ of this cross section. Physically, it is due to the charge-conjugation symmetry which exchanges the initial electron and positron.

As usual for annihilation processes in the ultra-relativistic limit, the cross-section (S.25) has divergent peaks in forward and backward directions, $\theta \rightarrow 0$ or $\theta \rightarrow \pi$. The divergence here is an artefact of the $m_e^2 = 0$ approximation, which becomes inaccurate at very small angles $\theta \lesssim (m_e/E)$ (or $\pi - \theta \lesssim (m_e/E)$). A more careful analysis shows that

for $\theta \lesssim \gamma^{-1}$,

$$\frac{d\sigma(e^-e^+\rightarrow \gamma S)}{d\Omega_{\text{c.m.}}} \approx \frac{\alpha g^2}{4\pi s} \times \left( \frac{s - M_s^2}{s} \times \frac{1}{\theta^2 + \gamma^{-2}} + \frac{M_s^2}{s - M_s^2} \times \frac{2\theta^2}{(\theta^2 + \gamma^{-2})^2} \right) \hfill (S.26)$$

(where $\gamma^{-1} = m_e/E \ll 1$) instead of eq. (S.25). Consequently, the total cross-section turns out to be finite rather than divergent, namely

$$\sigma_{\text{tot}}(e^-e^+\rightarrow \gamma S) = \alpha g^2 \times \frac{(s^2 + M_s^4)}{s^2(s - M_s^2)} \left( \log \frac{2E_e}{m_e} - \frac{sM_s^2}{s^2 + M_s^4} + O\left(\frac{m_e^2}{E_e^2}\right) \right). \hfill (S.27)$$
Problem 3(a):
The scalar potential part of the linear sigma model’s Lagrangian (2) is

\[ V(\phi) = \frac{\lambda}{8} \left( \sum_i \phi_i^2 - f^2 \right)^2 - \beta \lambda f^2 \times \phi_{N+1}, \]  

(S.28)

where the last term explicitly breaks the \(O(N+1)\) symmetry of the first term down to \(O(N)\).

To find the minimum of this potential, let’s first find the stationary points where all the first derivatives \(\partial V / \partial \phi_i\) are zero:

for \(i = 1, \ldots, N\), \(\frac{\partial V}{\partial \phi_i} = \frac{\lambda}{2} \left( \sum_j \phi_j^2 - f^2 \right) \times \phi_i = 0\),  

(S.29)

and \(\frac{\partial V}{\partial \phi_{N+1}} = \frac{\lambda}{2} \left( \sum_j \phi_j^2 - f^2 \right) \times \phi_{N+1} - \beta \lambda f^2 = 0\).  

(S.30)

From eq. (S.30) we immediately see that at any stationary point \(\sum \phi^2 - f^2 \neq 0\), hence eqs. (S.29) tell us that \(\phi_1 = \cdots = \phi_N = 0\). In other words, all the stationary points lie on the \(\phi_{N+1}\) axis in the \((N+1)\) dimensional space of the scalar field values. And in this space, eq. (S.30) becomes a simple cubic equation

\[ \phi_{N+1}^3 - f^2 \times \Phi_{N+1} - 2\beta f^2 = 0. \]  

(S.31)

For small \(\beta \ll f\), this cubic equation has 3 real solutions, approximately

\[ \langle \phi_{N+1} \rangle_1 \approx -2\beta, \quad \langle \phi_{N+1} \rangle_2 \approx -f + \beta, \quad \langle \phi_{N+1} \rangle_3 \approx +f + \beta. \]  

(S.32)

Now let’s find out which of the three stationary points is a minimum (or at least a local minimum) by looking at the second derivatives of the potential (S.28). Along the \(\phi_{N+1}\) axis in the field space, the second derivatives amount to

\[ \frac{\partial^2 V}{\partial \phi_i \partial \phi_j} = \frac{\lambda}{2} \times \begin{cases} (3\phi_{N+1}^2 - f^2) & \text{for } i = j = N + 1, \\ 0 & \text{for } i \leq N, \ j = N + 1 \text{ or } j \leq N, \ i = N + 1, \\ (\phi_{N+1}^2 - f^2) \times \delta_{ij} & \text{for } i, j \leq N. \end{cases} \]  

(S.33)

Evaluating these derivatives for the 3 stationary points (S.32) — while assuming small \(\beta > 0\)
— gives us

\[
\begin{align*}
@ \langle \phi_{N+1} \rangle_1 : & \quad \frac{\partial^2 V}{(\partial \phi_{N+1})^2} < 0 \quad \text{while other} \quad \frac{\partial^2 V}{(\partial \phi_i)^2} < 0 \quad \Longrightarrow \quad \text{maximum}, \\
@ \langle \phi_{N+1} \rangle_2 : & \quad \frac{\partial^2 V}{(\partial \phi_{N+1})^2} > 0 \quad \text{while other} \quad \frac{\partial^2 V}{(\partial \phi_i)^2} < 0 \quad \Longrightarrow \quad \text{saddle point}, \\
@ \langle \phi_{N+1} \rangle_3 : & \quad \frac{\partial^2 V}{(\partial \phi_{N+1})^2} > 0 \quad \text{while other} \quad \frac{\partial^2 V}{(\partial \phi_i)^2} > 0 \quad \Longrightarrow \quad \text{minimum}.
\end{align*}
\]

Thus, the potential (S.28) has a unique minimum at

\[
\langle \phi_1 \rangle = \cdots = \langle \phi_N \rangle = 0, \quad \langle \phi_{N+1} \rangle = +f + \beta + O(\beta^2/f).
\]

(S.34)

\textit{Quod erat demonstrandum.}

\underline{Problem 3(b):}

Let’s shift the fields as in eq. (4). In terms of the shifted fields,

\[
T \stackrel{\text{def}}{=} \sum_i \phi_i^2 - f^2 = \bar{\pi}^2 + (\sigma + \langle \phi_{N+1} \rangle)^2 - f^2 = \bar{\pi}^2 + \sigma^2 + 2 \langle \phi_{N+1} \rangle \times \sigma + (\langle \phi_{N+1} \rangle^2 - f^2),
\]

(S.35)

where \( \bar{\pi} \) is a short-hand for \( N\)-vector \( (\pi^1, \ldots, \pi^N) \) of the pion fields, thus \( \bar{\pi}^2 = (\pi^1)^2 + \cdots + (\pi^N)^2 \).

Therefore, expanding the scalar potential (S.28) into powers of the shifted fields, we obtain

\[
V = \frac{\lambda}{8} \times T^2 - \beta \lambda f^2 \times (\sigma + \langle \phi_{N+1} \rangle)
\]

\[
= \frac{\lambda}{8} \times (\bar{\pi}^2 + \sigma^2)^2 + \frac{\lambda \langle \phi_{N+1} \rangle^2}{2} \times \sigma \times (\bar{\pi}^2 + \sigma^3)
\]

\[
+ \frac{\lambda \langle \phi_{N+1} \rangle^2}{2} \times \sigma^2 + \frac{\lambda (\langle \phi_{N+1} \rangle^2 - f^2)}{4} \times (\bar{\pi}^2 + \sigma^2)
\]

\[
+ \left( \frac{\lambda \langle \phi_{N+1} \rangle^2}{2} \right) \times (\langle \phi_{N+1} \rangle^2 - f^2) - \beta \lambda f^2 \right) \times \sigma + \text{const}.
\]

(S.36)

On the last line here, the coefficient of \( \sigma \) vanishes thanks to \( \langle \phi_{N+1} \rangle \) obeying the cubic equation (S.31). For the same reason, the coefficient of \( (\bar{\pi}^2 + \sigma^2) \) on the line before the last may be
simplified as
\[ \frac{\lambda (\langle \phi_{N+1} \rangle^2 - f^2)}{4} = \frac{\beta \lambda f^2}{2 \langle \phi_{N+1} \rangle}. \]  

(S.37)

Altogether, we have
\[ V(\sigma, \pi) = \frac{\lambda}{8} \times (\pi^2 + \sigma^2)^2 + \frac{\mu}{2} \times (\sigma^3 + \sigma \pi^2) + \frac{M^2_\sigma}{2} \times \sigma^2 + \frac{M^2_\pi}{2} \times \pi^2 + \text{const}, \]  

(S.38)

where

- quartic coupling \( \lambda = \lambda \),
- cubic coupling \( \mu = \lambda \times \langle \phi_{N+1} \rangle \approx \lambda (f + \beta) \),
- pion mass \( M^2_\pi = \frac{\beta \lambda f^2}{\langle \phi_{N+1} \rangle} \approx \beta \lambda f \),
- sigma mass \( M^2_\sigma = M^2_\pi + \lambda \langle \phi_{N+1} \rangle^2 \approx \lambda f (f + 3 \beta) \).

(S.39)

Let’s take a closer look at the pion’s mass\(^2\), \( M^2_\pi \approx \beta \times \lambda f \). In the \( \beta = 0 \) limit, the pions are massless in accordance with the Goldstone theorem. Indeed, for \( \beta = 0 \) the sigma model’s Lagrangian has exact \( SO(N+1) \) symmetry which is spontaneously broken down to \( SO(N) \); there are \( N \) spontaneously broken generators, so there should be \( N \) massless Goldstone bosons. But for \( \beta \neq 0 \), the \( SO(N+1) \) symmetry of the Lagrangian is only approximate, and its explicit breaking by the \( \beta \lambda f^2 \times \phi_{N+1} \) term spoils the Goldstone theorem. Thus, instead of exactly massless Goldstone bosons we should get light but not quite massless pseudo-Goldstone bosons; to the first order in \( \beta \), their mass\(^2\) should be proportional to \( \beta \). And indeed, in the linear sigma model \( M^2_\pi \approx \beta \times \lambda f \).

Still, for \( \beta \ll f \), the pions should be much lighter than the sigma particle. And indeed, according to eqs. (S.39),
\[ \frac{M^2_\pi}{M^2_\sigma} \approx \frac{\beta \lambda f}{\lambda f^2} = \frac{\beta}{f} \ll 1. \]  

(S.40)
Problem 3(c): Back in homework#9 (problem 3), we had a very similar setup to the *shifted* fields of the linear sigma models: $N + 1$ scalar fields $\sigma(x)$ and $\pi^i(x)$, with the Lagrangian

$$
\mathcal{L} = \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{1}{2} (\partial_\mu \pi)^2 - V,
$$

$$
V(\sigma, \pi) = \frac{\lambda}{8} (\pi^2 + \sigma^2 + 2f \times \sigma)^2 - \lambda \times (\sigma^2 + \pi^2)^2. \tag{9.3}
$$

In particular, there is a mass term for the $\sigma$ field but not for the pions, which are exactly massless — exactly as in the present sigma model with $\beta = 0$. The cubic and quartic terms in the potential (9.3) also have exactly the same form as in eq. (S.38), and the couplings $\lambda$ and $\mu = \lambda f$ are related to the $\sigma$ field’s mass as

$$
\mu^2 = \lambda \times M_\sigma^2. \tag{S.41}
$$

For the present sigma model, we have exactly similar relation for $\beta = 0$. Indeed, according to eq. (S.39),

$$
\text{for } \beta = 0, \quad \mu = \lambda f, \quad M_\sigma^2 = \lambda f^2 \quad \implies \quad \mu^2 = \lambda \times M_\sigma^2. \tag{S.42}
$$

Therefore, the $\pi \pi \to \pi \pi$ scattering amplitudes in the linear sigma model for $\beta = 0$ come out to be exactly as homework#9. At the tree level,

$$
\mathcal{M}(\pi^j + \pi^k \to \pi^\ell + \pi^m) = -\left(\lambda + \frac{\mu^2}{s - M_\sigma^2}\right) \times \delta^k \delta^\ell \delta^m - \left(\lambda + \frac{\mu^2}{t - M_\sigma^2}\right) \times \delta^j \delta^\ell \delta^m
$$

$$
- \left(\lambda + \frac{\mu^2}{u - M_\sigma^2}\right) \times \delta^j \delta^k \delta^m,
$$

which in light of the relation (S.41) becomes

$$
\mathcal{M}(\pi^j + \pi^k \to \pi^\ell + \pi^m) = -\frac{\lambda s}{s - M_\sigma^2} \times \delta^j \delta^k \delta^m - \frac{\lambda t}{t - M_\sigma^2} \times \delta^j \delta^\ell \delta^m
$$

$$
- \frac{\lambda u}{u - M_\sigma^2} \times \delta^j \delta^k \delta^m. \tag{S.44}
$$

When any of the 4 pions’ energy becomes small, we get $s, t, u \ll M_\sigma^2$, and the scattering
amplitude becomes small as

\[
\mathcal{M} \approx \frac{\lambda}{M^2_\sigma} \times \left( s \times \delta^{jk} \delta^{\ell m} + t \times \delta^{j\ell} \delta^{km} + u \times \delta^{jm} \delta^{k\ell} \right) = O\left( \frac{\lambda E_{\text{cm}}^2}{M^2_\sigma} \right). \tag{S.45}
\]

Problem 3(d): For \( \beta \neq 0 \), the quartic and the cubic couplings of the \( \sigma \) and \( \pi^i \) to each other has similar overall form to what we had back in homework\#9, but the overall coefficients \( \lambda \) and \( \mu \) of those couplings are no longer related to the \( \sigma \) particle’s mass by eq. (S.41). Instead, eq. (S.39) gives us

\[
\mu = \lambda \langle \phi_{N+!} \rangle, \quad M^2_\sigma = M^2_\pi + \lambda \langle \phi_{N+!} \rangle^2 \quad \Rightarrow \quad \mu^2 = \lambda \times (M^2_\sigma - M^2_\pi). \tag{S.46}
\]

Now consider the \( \pi \pi \to \pi \pi \) scattering. At the tree level, we have exactly the same 4 diagrams for such scattering as in the homework\#9, namely

\[
\begin{align*}
\pi^j(p_1) & \quad \pi^\ell(p'_1) & \quad \pi^j(p_1) & \quad \pi^\ell(p'_1) \\
\pi^k(p_2) & \quad \pi^m(p'_2) & \quad \pi^k(p_2) & \quad \pi^m(p'_2)
\end{align*}
\]

\[
\begin{align*}
\pi^j(p_1) & \quad \pi^\ell(p'_1) & \quad \pi^j(p_1) & \quad \pi^\ell(p'_1) \\
\pi^k(p_2) & \quad \pi^m(p'_2) & \quad \pi^k(p_2) & \quad \pi^m(p'_2)
\end{align*}
\]

Altogether, these diagrams yield the scattering amplitude exactly as in eq. (S.43), but for the
\[
\lambda + \frac{\mu^2}{s - M^2_{\sigma}} = \frac{\lambda s - \lambda M^2_{\sigma} + \mu^2}{s - M^2_{\sigma}} = \frac{\lambda s - \lambda M^2_{\pi}}{s - M^2_{\sigma}} \tag{S.48}
\]

and likewise
\[
\lambda + \frac{\mu^2}{t - M^2_{\sigma}} = \frac{\lambda(t - M^2_{\pi})}{t - M^2_{\sigma}} \quad \text{and} \quad \lambda + \frac{\mu^2}{u - M^2_{\sigma}} = \frac{\lambda(u - M^2_{\pi})}{u - M^2_{\sigma}}, \tag{S.49}
\]

so the amplitude (S.43) becomes
\[
\mathcal{M}(\pi^j + \pi^k \rightarrow \pi^\ell + \pi^m) = -\frac{\lambda(s - M^2_{\pi})}{s - M^2_{\sigma}} \times \delta^{jk}\delta^{\ell m} - \frac{\lambda(t - M^2_{\pi})}{t - M^2_{\sigma}} \times \delta^{j\ell}\delta^{km} - \frac{\lambda(u - M^2_{\pi})}{u - M^2_{\sigma}} \times \delta^{jm}\delta^{k\ell}. \tag{S.50}
\]

When the pions’ energies become low compared to \(M_{\sigma}\) — or in Lorentz-invariant terms, when \(s, t, u \ll M^2_{\sigma}\) — we may simplify this amplitude by approximating all the denominators as \(-M^2_{\sigma}\), thus
\[
\mathcal{M}(\pi^j + \pi^k \rightarrow \pi^\ell + \pi^m) \approx \left(\frac{\lambda}{M^2_{\sigma}} \approx \frac{1}{f^2}\right) \times \left(\frac{1}{s - M^2_{\pi}} \times \delta^{jk}\delta^{\ell m} \times (s - M^2_{\pi}) + \frac{1}{t - M^2_{\pi}} \times \delta^{j\ell}\delta^{km} \times (t - M^2_{\pi}) + \frac{1}{u - M^2_{\pi}} \times \delta^{jm}\delta^{k\ell} \times (u - M^2_{\pi})\right). \tag{5}
\]

What happens to this amplitude when one of the pions’ momentum becomes very small? Alas, for \(\beta \neq 0\) the pions are massive, so we cannot take all 4 components of a pion’s \(p^\mu\) to zero. The best we can do is to take \(\mathbf{p} \rightarrow 0\) while \(p^0 \rightarrow m\), which is the non-relativistic limit. However, if only one pion is non-relativistic while the other 3 pions have \(E \gg M_{\pi}\) (but \(E \ll M_{\sigma}\)), we generally have \(s, t, u = O(E \times M_{\pi}) \gg M^2_{\pi}\) (although \(s, t, u \ll M^2_{\sigma}\)), and the scattering amplitude becomes
\[
\mathcal{M} = O\left(\frac{E \times M_{\pi}}{f^2}\right) \not\rightarrow 0. \tag{S.51}
\]

The strongest low-energy limit we can take for massive pions is to make all four pions non-relativistic. In this limit, \(s = E^2_{\text{cm}} \approx 4M^2_{\pi}\) while \(u, t = O(p^2) \ll M^2_{\pi}\), so the scattering
amplitude (5) becomes

\[ \mathcal{M}(\pi^j + \pi^k \to \pi^\ell + \pi^m) \approx \left( \frac{\lambda M^2}{M_\sigma^2} \approx \frac{\beta \lambda}{f} \right) \times (3\delta^{jk}\delta^{\ell m} - \delta^{j\ell}\delta^{km} - \delta^{jm}\delta^{kl}). \]  

(6)

This amplitude is suppressed by the factor \( \beta/f \), but it does not vanish! And even if all 4 pions belong to the same species, the scattering amplitude does not vanish in the non-relativistic limit,

\[ \mathcal{M}(\pi^1 + \pi^1 \to \pi^1 + \pi^1) \approx \frac{\lambda \beta}{f} \neq 0, \quad (S.52) \]

unlike what we had in homework #9 in the low-energy limit for the massless pions.

**Problem 4(a):**

given \( \Phi \to e^{+\theta} U_L \Phi U_R^\dagger \),  

we have \( \Phi^\dagger \to e^{-\theta} U_R \Phi^\dagger U_L^\dagger \),  

hence \( \Phi^\dagger \Phi \to U_R (\Phi^\dagger \Phi) U_R^\dagger \),  

(\( \Phi^\dagger \Phi \))^2 \( \to U_R \Phi^\dagger U_R^\dagger U_R \Phi^\dagger U_R^\dagger = U_R (\Phi^\dagger \Phi)^2 U_R^\dagger \),  

likewise \( (\Phi^\dagger \Phi)^n \to U_R (\Phi^\dagger \Phi)^n U_R^\dagger \forall n = 1, 2, 3, \ldots, \)  

(S.56)

and therefore

all traces \( \text{tr} \left( (\Phi^\dagger \Phi)^n \right) \) are invariant under symmetries (9),  

(S.57)

thanks to the cyclic invariance rule for traces, \( \text{tr}(U_R X U_R^\dagger) = \text{tr}(X U_R^\dagger U_R) = \text{tr}(X) \) for any \( X = (\Phi^\dagger \Phi)^n \). Consequently, the scalar potential (8) is invariant under symmetries (9).

For the global symmetries where \( e^{i\theta}, U_L, \) and \( U_R \) do not depend on \( x \), the kinetic term
in (7) is also invariant. Indeed,

\[ \begin{align*}
\partial_\mu \Phi & \to e^{i\theta} U_L (\partial_\mu \Phi) U_R^\dagger, \\
\partial_\mu \Phi^\dagger & \to e^{-i\theta} U_R (\partial_\mu \Phi^\dagger) U_L^\dagger, \\
\partial^\mu \Phi^\dagger \partial_\mu \Phi & \to U_R (\partial^\mu \Phi^\dagger \partial_\mu \Phi) U_L^\dagger,
\end{align*} \]

and \( \text{tr}(\partial^\mu \Phi^\dagger \partial_\mu \Phi) \) is invariant.

Altogether, the whole Lagrangian (7) is invariant, \( \mathcal{Q.E.D.} \).

**Problem 4(\star):**

The kinetic term in (7) and the last two terms in the potential (8) have a much bigger symmetry than \( G = SU(N) \times SU(N) \times U(1) \), namely \( SO(2N^2) \) which does not care for the matrix structure of the \( \Phi(x) \) and treats it as \( 2N^2 \) real component fields. Indeed,

\[ \text{tr}(\Phi^\dagger \Phi) = \sum_{i,j} |\Phi_i^j|^2 = \sum_{i,j} \left( (\text{Re} \Phi_i^j)^2 + (\text{Im} \Phi_i^j)^2 \right) \]

is invariant under all \( SO(2N^2) \) “rotations” of the components, and so is the kinetic term.

On the other hand, the \( \text{tr}(\Phi^\dagger \Phi \Phi^\dagger \Phi) \) in the potential does depend on the packing of \( 2N^2 \) real components into a complex \( N \times N \) matrix, and it is this term which reduces the internal symmetry group of the theory to \( G = SU(N) \times SU(N) \times U(1) \).

Proving that all the \( SO(2N^2)/G \) symmetries are broken by the quartic trace term is a non-trivial exercise in group theory rather than field theory. You do not have to do it as part of this homework set, and I am not writing down the proof here.

**Problem 4(b):**

Given the eigenvalues \( (\kappa_1, \ldots, \kappa_N) \) of the \( \Phi^\dagger \Phi \) matrix, the invariant traces (S.57) obtain as

\[ \text{tr} \left( (\Phi^\dagger \Phi)^n \right) = \sum_{i=1}^{N} \kappa_i^n. \]
Consequently, the scalar potential is

\[ V = \frac{\alpha}{2} \sum_i \kappa_i^2 + \frac{\beta}{2} \left( \sum_i \kappa_i \right)^2 + m^2 \sum_i \kappa_i. \quad \text{(S.61)} \]

Now let’s minimize this potential. Since the matrix $\Phi^\dagger \Phi$ cannot have any negative eigenvalues, we are looking for a minimum of $V(\kappa_1, \ldots, \kappa_N)$ under constraints $\kappa_i \geq 0$. This requires

\[ \forall i = 1, \ldots, N, \quad \text{either } \kappa_i \geq 0 \text{ and } \frac{\partial V}{\partial \kappa_i} = 0, \quad \text{or else } \kappa_i = 0 \text{ and } \frac{\partial V}{\partial \kappa_i} > 0, \quad \text{(S.62)} \]

where

\[ \frac{\partial V}{\partial \kappa_i} = \alpha \kappa_i + m^2 + \beta \sum_j \kappa_j. \quad \text{(S.63)} \]

These derivatives are linear functions of the eigenvalues $\kappa_i$, so all the non-zero eigenvalues must obey the same linear equation

\[ \alpha \times \kappa_i = -m^2 - \times \sum_j \kappa_j, \quad \text{same for all } \kappa_i \neq 0, \]

which means that all non-zero $\kappa_i$ have the same value. Thus, up to a permutation of eigenvalues,

\[ \kappa_1 = \cdots = \kappa_k = C^2, \quad \kappa_{k+1} = \cdots = \kappa_N = 0, \quad \text{(S.64)} \]

for some $k = 0, 1, 2, \ldots, N$, and $C^2$ obtains from

\[ \alpha \times C^2 + m^2 + \beta \times kC^2 = 0 \quad \rightarrow \quad C^2 = \frac{-m^2}{\alpha + k\beta}. \quad \text{(S.65)} \]

To make sure that the solution (S.64) is a minimum rather than a maximum or a saddle point, we need

\[ C^2 = \frac{-m^2}{\alpha + k\beta} > 0 \quad \text{unless } k = 0, \quad \text{(S.66)} \]

\[ m^2 + \beta kC^2 = \frac{\alpha m^2}{\alpha + k\beta} > 0 \quad \text{unless } k = N. \]

Depending on the signs of $\alpha$, $\beta$ and $m^2$ parameters, this limits the solutions to the following:
For $\alpha > 0$, $\beta > 0$, and $m^2 > 0$, the only solution is $k = 0$, which means $\kappa_1 = \cdots \kappa_N = 0$ and hence $\langle \Phi \rangle = 0$.

For $\alpha > 0$, $\beta > 0$, and $m^2 < 0$, the only solutions is $k = N$, which means

$$\kappa_1 = \cdots = \kappa_N = C^2 = \frac{-m^2}{\alpha + N\beta} > 0,$$

and hence $\langle \Phi \rangle = C \times a \text{unitary matrix.}$ We shall focus on this regime through the rest of this problem.

For $\alpha < 0$ or $\beta < 0$, the situation is more complicated:

— When $\alpha + \beta < 0$ or $\alpha + N\beta < 0$, the scalar potential (8) is unbounded from below and the theory is sick.

— When $\alpha > 0$ and $\beta < 0$ but $\alpha + N\beta > 0$, the solutions are similar to the $\beta > 0$ case: For $m^2 > 0$ all $\kappa_i = 0$, while for $m^2 < 0$ the $\kappa_i$ are as in eq. (10).

— When $\beta > 0$ and $\alpha < 0$ but $\alpha + \beta > 0$: for $m^2 > 0$ the only solution is $k = 0$, meaning $\langle \Phi \rangle = 0$, but for $m^2 < 0$ all the solutions (S.64) with $k = 1, 2, \ldots, N$ are good local minima.

To find the global minimum, we compare the potentials at the local minima,

$$V(\text{minimum#}k) = \frac{\alpha}{2} \times kC^4 + \frac{\beta}{2} \times (kC^2)^2 + m^2 \times kC^2$$

$$= \frac{k\alpha + k^2\beta}{2} \times \frac{m^4}{(\alpha + k\beta)^2} + km^2 \times \frac{-m^2}{(\alpha + k\beta)}$$

$$= \frac{-m^4}{2} \times \frac{k}{k\beta + \alpha}.$$

Since $\alpha < 0$ but $\alpha + \beta > 0$, the deepest minimum obtains for $k = 1$, thus

$$\kappa_1 = \frac{-m^2}{\alpha + \beta}, \quad \kappa_2 = \cdots = \kappa_N = 0.$$

(S.68)
Problem 4(c):  
Let’s act with some $SU(N)_L \times SU(N)_R \times U(1)$ symmetry (9) on the vacuum expectation values (11):

$$\langle \Phi \rangle = C \times 1_{N \times N} \rightarrow e^{i\theta} U_L \langle \Phi \rangle U_R^\dagger = C \times e^{i\theta} U_L U_R^\dagger.$$  \hfill (S.69)

Clearly, to keep the VEVs $\langle \Phi \rangle$ invariant, we need

$$e^{i\theta} U_L U_R^\dagger = 1_{N \times N}$$ \hfill (S.70)

and hence

$$U_R = e^{i\theta} \times U_L.$$ \hfill (S.71)

Moreover, since the $U_L$ and $U_R$ matrices have unit determinants, this requires

$$\text{det} \left( e^{i\theta} \times 1_{N \times N} \right) = 1 \implies N \times \theta = 0 \pmod{2\pi}.$$ \hfill (S.72)

Such a phase can be absorbed into the $U_L \in SU(N)$, so without loss of generality we need

$$e^{i\theta} = 1 \quad \text{and} \quad U_L = U_R \in SU(N).$$ \hfill (S.73)

In other words, the unbroken symmetry group is $SU(N)$ which acts on the scalar fields as

$$\Phi(x) \rightarrow U \Phi(x) U^\dagger, \quad U \in SU(N).$$ \hfill (S.74)
Problem 4(d):

In terms of the shifted fields (12),

\[ \partial_\mu \Phi = \frac{1}{\sqrt{2}} (\partial_\mu \varphi_1 + i \partial_\mu \varphi_2), \quad \partial_\mu \Phi^\dagger = \frac{1}{\sqrt{2}} (\partial_\mu \varphi_1 - i \partial_\mu \varphi_2), \quad (S.75) \]

hence the kinetic term in the Lagrangian becomes

\[ \text{tr} \left( \partial_\mu \Phi^\dagger \partial^\mu \Phi \right) = \frac{1}{2} \text{tr} (\partial_\mu \varphi_1 \partial^\mu \varphi_1) + \frac{1}{2} \text{tr} (\partial_\mu \varphi_2 \partial^\mu \varphi_2). \quad (S.76) \]

As to the potential terms, we have

\[ \Phi^\dagger \Phi = C^2 \times 1_{N \times N} + C \times (\delta \Phi^\dagger + \delta \Phi) + \delta \Phi^\dagger \delta \Phi \]
\[ = C^2 \times 1_{N \times N} + \sqrt{2} C \times \varphi_1 + \frac{1}{2} \varphi_1^2 + \frac{1}{2} \varphi_2^2 + \frac{i}{2} [\varphi_1, \varphi_2] \quad (S.77) \]

and consequently

\[ \text{tr} (\Phi^\dagger \Phi) = NC^2 + \sqrt{2} C \text{tr} (\varphi_1) + \frac{1}{2} \text{tr} (\varphi_1^2) + \frac{1}{2} \text{tr} (\varphi_2^2) \quad (S.78) \]
\[ \text{tr}^2 (\Phi^\dagger \Phi) = N^2 C^4 + 2 \sqrt{2} N C^3 \text{tr} (\varphi_1) + 2 C^2 \text{tr}^2 (\varphi_1) + NC^2 (\text{tr} (\varphi_1^2) + \text{tr} (\varphi_2^2)) + \sqrt{2} C \text{tr} (\varphi_1) \times (\text{tr} (\varphi_1^2) + \text{tr} (\varphi_2^2)) + \frac{1}{4} (\text{tr} (\varphi_1^2) + \text{tr} (\varphi_2^2))^2 \quad (S.79) \]
\[ \text{tr} \left( (\Phi^\dagger \Phi)^2 \right) = NC^4 + 2 \sqrt{2} C^3 \text{tr} (\varphi_1) + 3 C^2 \text{tr} (\varphi_1^2) + C^2 \text{tr} (\varphi_2^2) + \frac{\sqrt{2} C \text{tr} (\varphi_1^3)}{2} + \frac{\sqrt{2} C \text{tr} (\varphi_1 \varphi_2^2)}{2} + \frac{1}{4} \text{tr} (\varphi_1^4) + \frac{1}{4} \text{tr} (\varphi_2^4) + \frac{3}{2} \text{tr} (\varphi_1^2 \varphi_2^2) - \frac{1}{2} \text{tr} (\varphi_1 \varphi_2 \varphi_1 \varphi_2). \quad (S.80) \]

Plugging all these formulae into the potential (8) and expanding in powers of \( \varphi_1 \) and \( \varphi_2 \), we obtain

\[ V = \text{const} + V_1 + V_2 + V_3 + V_4 \quad (S.81) \]

where

\[ V_1 = \sqrt{2} C \times (m^2 + \beta NC^2 + \alpha C^2) \times \text{tr} (\varphi_1) = 0 \quad (S.82) \]

because \( m^2 + (\alpha + N \beta) C^2 = 0 \).
\[ V_2 = \beta C^2 \times \text{tr}^2(\varphi_1) + \frac{1}{2}(m^2 + \beta NC^2 + 3\alpha C^2) \times \text{tr}(\varphi_1^2) \]
\[ + \frac{1}{2}(m^2 + \beta NC^2 + \alpha C^2) \times \text{tr}(\varphi_2^2) \]
\[ = \beta C^2 \times \text{tr}^2(\varphi_1) + \alpha C^2 \text{tr}(\varphi_1^2) + 0, \quad \text{(S.83)} \]

\[ V_3 = \frac{\beta C}{\sqrt{2}} \times \text{tr}(\varphi_1) \times \left( \text{tr}(\varphi_1^2) + \text{tr}(\varphi_2^2) \right) + \frac{\alpha C}{\sqrt{2}} \times \left( \text{tr}(\varphi_1^3) + \text{tr}(\varphi_1 \varphi_2^2) \right), \quad \text{(S.84)} \]

\[ V_4 = \frac{\beta}{8} \left( \text{tr}(\varphi_1^2) + \text{tr}(\varphi_2^2) \right)^2 \]
\[ + \frac{\alpha}{8} \left( \text{tr}(\varphi_1^4) + \text{tr}(\varphi_2^4) + 6 \text{tr}(\varphi_1^2 \varphi_2^2) - 2 \text{tr}(\varphi_1 \varphi_2 \varphi_1 \varphi_2) \right), \quad \text{(S.85)} \]

Combining the quadratic part (S.83) of this potential with the kinetic terms (S.76), we arrive at

\[ \mathcal{L}_2 = \frac{1}{2} \text{tr}(\partial_{\mu} \varphi_1 \partial^{\mu} \varphi_1) - \beta C^2 \times \text{tr}^2(\varphi_1) - \alpha C^2 \text{tr}(\varphi_1^2) + \frac{1}{2} \text{tr}(\partial_{\mu} \varphi_2 \partial^{\mu} \varphi_2) \quad \text{(S.86)} \]

which gives us the mass spectrum of the theory: the \( \varphi_1(x) \) matrix of fields is massive and the \( \varphi_2(x) \) matrix is massless. Each matrix is \( N \times N \) and hermitian, so it contains \( N^2 \) independent real scalar fields, which give rise to \( N^2 \) particles. Altogether, the spectrum comprises:

- \( N^2 \) massless particles from the \( \varphi_2(x) \) matrix.
- \( N^2 - 1 \) massive particles with \( M^2 = 2\alpha C^2 \) from the traceless part of the \( \varphi_1(x) \) matrix.
- One more massive particle with \( M^2 = 2(\alpha + N\beta)C^2 = -2m^2 \) from the trace of \( \varphi_1(x) \).

To see where the values of the masses come from, let’s decompose the \( \varphi_1 \) matrix into the pure trace plus the traceless part,

\[ \xi(x) \overset{\text{def}}{=} \frac{\text{tr}(\varphi_1(x))}{\sqrt{N}} \quad \text{and} \quad \tilde{\varphi}_1(x) \overset{\text{def}}{=} \varphi_1(x) - \frac{\xi(x)}{\sqrt{N}} \times \mathbf{1}_{N \times N} \quad \implies \text{tr}(\tilde{\varphi}_1) \equiv 0. \quad \text{(S.87)} \]

Consequently,

\[ \text{tr}^2(\varphi_1) = N \times \xi^2, \quad \text{tr}(\varphi_1^2) = \xi^2 + \text{tr}(\tilde{\varphi}_1^2), \quad \text{(S.88)} \]
likewise

$$\text{tr}(\partial_\mu \varphi_1 \partial^\mu \varphi_1) = \partial_\mu \xi \partial^\mu \xi + \text{tr}(\partial_\mu \tilde{\varphi}_1 \partial^\mu \tilde{\varphi}_1),$$

so the free Lagrangian (S.86) becomes

$$L_2 = \frac{1}{2}(\partial_\mu \xi)^2 - 2(\beta C^2 N + \alpha C^2) \times \frac{1}{2} \xi^2 + \frac{1}{2} \text{tr}((\partial_\mu \tilde{\varphi}_1)^2) - 2\alpha C^2 \times \frac{1}{2} \text{tr}(\tilde{\varphi}_1^2) + \frac{1}{2} \text{tr}((\partial_\mu \varphi_2)^2) - 0 \times \frac{1}{2} \text{tr}(\varphi_2^2) \quad (S.89)$$

where all the masses are manifest.

Problem 4(e):
The unbroken $SU(N)$ symmetry acts on the scalar fields according to

$$\Phi(x) \rightarrow U \times \Phi(x) \times U^\dagger. \quad (S.74)$$

and since the VEV (11) is invariant, the shifted fields $\delta \Phi(x) = \Phi(x) \langle \Phi \rangle$ also transform according to

$$\delta \Phi(x) \rightarrow U \times \delta \Phi(x) \times U^\dagger. \quad (S.90)$$

Moreover, unitary transforms like these preserve hermiticity, so when we decompose $\delta \Phi(x)$ into a hermitian matrix $\varphi_1(x)$ and an antihermitean matrix $i\varphi_2(x)$, the transforms (S.90) do not mix the $\varphi_1$ and $\varphi_2$ with each other. Instead, they transform like

$$\varphi_1(x) \rightarrow U \varphi_1(x) U^\dagger, \quad \varphi_2(x) \rightarrow U \varphi_2(x) U^\dagger, \quad (S.91)$$

which means that $\varphi_1$ and $\varphi_2$ comprise separate $SU(N)$ multiplets. Furthermore, the transforms (S.91) preserve trances $\text{tr}(\varphi_1)$ and $\text{tr}(\varphi_2)$, so to make the $SU(N)$ multiplet structure manifest, let’s decompose both $\varphi_1$ and $\varphi_2$ into their traceless parts and pure traces along the
lines of eq. (S.87),

\[ \varphi_1(x) = \frac{\xi_1(x)}{\sqrt{N}} \times 1_{N \times N} + \tilde{\varphi}_1(x), \quad \varphi_2(x) = \frac{\xi_2(x)}{\sqrt{N}} \times 1_{N \times N} + \tilde{\varphi}_2(x), \quad \text{tr}(\tilde{\varphi}_1) \equiv \text{tr}(\tilde{\varphi}_2) \equiv 0. \]

(S.92)

With this decomposition, the \( \xi_1 \) and the \( \xi_2 \) are both invariant under the \( SU(N) \) — which puts each of them into its own singlet multiplet — while the each of the traceless parts \( \tilde{\varphi}_1 \) and \( \tilde{\varphi}_2 \) makes each own adjoint multiplet.

This multiplet structure agrees with the masses we obtained in part (d). Indeed, all \( N^2 - 1 \) members of the adjoint multiplet \( \tilde{\varphi}_1 \) have the same mass \( 2\alpha C^2 \), while the singlet \( \xi_1 \) has a different mass \( 2(\alpha + N\beta)C^2 \).

On the other hand, both the adjoint multiplet \( \tilde{\varphi}_2 \) and the singlet \( \xi_2 \) are massless. The reason for this degeneracy goes beyond the un-broken \( SU(N) \) symmetry; instead, both the \( \tilde{\varphi}_2 \) and the \( \xi_2 \) are Goldstone bosons of the spontaneously broken symmetries in

\[ G/H = \left( SU(N)_L \times SU(N)_R \times U(1) \right) / SU(N). \]

(S.93)

Specifically, the singlet \( \xi_2 \) is the Goldstone boson of the broken \( U(1) \) symmetry. Indeed, the \( U(1) \)'s generator commutes with all the other generators, so it belongs in its own singlet of the symmetry, and the corresponding Goldstone particle should also be a singlet.

Now consider the non-abelian generators. Generators \( T^a_L \) of the \( SU(N)_L \) form an adjoint multiplet of the \( SU(N)_L \), but are invariant under the \( SU(N)_R \). Likewise, generators \( T^a_R \) of the \( SU(N)_R \) form an adjoint multiplet of the \( SU(N)_R \), but are invariant under the \( SU(N)_L \). In other words, under an \( (U_L, U_R) \in SU(N)_L \times SU(N)_R \) they transform as

\[ T^a_L \rightarrow U_L T^a_L U_L^\dagger, \quad T^a_R \rightarrow U_R T^a_R U_R^\dagger. \]

(S.94)

When the \( SU(N)_L \times SU(N)_R \) is broken down to a single \( SU(N) \) spanning \( U_L = U_R = U \), both \( T^a_L \) and \( T^a_R \) transform as

\[ T^a_L \rightarrow UT^a_L U^\dagger, \quad T^a_R \rightarrow UT^a_R U^\dagger, \]

(S.95)

which puts them into two adjoint multiplets of the unbroken \( SU(N) \). Equivalently, we may
form two adjoint multiplets out of

$$T^a_V = T^a_L + T^a_R$$ and $$T^a_A = T^a_L - T^a_R,$$ \hspace{1cm} \text{(S.96)}

which act on the scalar fields according to

$$T^a_V \Phi = \frac{i}{2}[\lambda^a, \Phi], \quad T^a_A \Phi = \frac{i}{2}\{\lambda^a, \Phi\}. \hspace{1cm} \text{(S.97)}$$

The $T^a_V$ generate the unbroken $SU(N)$ symmetry, cf. eq. (S.74). The $T^a_A$ generators are spontaneously broken, hence there should be an adjoint multiplet of massless Goldstone bosons. And indeed there is — the \(\tilde{\varphi}_2\).