Problem 1(a):
In $N \times N$ matrix form, the local $SU(N)$ symmetry acts on the adjoint matter field $\Phi(x)$ and the gauge field $A_\mu(x)$ according to

\[ \Phi'(x) = U(x)\Phi(x)U^\dagger(x), \quad A'_\mu(x) = U(x)A_\mu(x)U^\dagger(x) + i\partial_\mu U(x)U^\dagger(x). \]  

(S.1)

Consequently, the covariant derivatives (4) become

\[ D_\mu \Phi(x) \rightarrow D'_\mu \Phi'(x) = \partial_\mu \Phi(x) + i [A'_\mu(x), \Phi'(x)] \]  

(S.2)

where the first term on the RHS expands to

\[
\partial_\mu \Phi' = \partial_\mu (U\Phi U^\dagger) \\
= (\partial_\mu U)\Phi U^\dagger + U(\partial_\mu \Phi)U^\dagger + U\Phi(\partial_\mu U^\dagger) \\
= UU^\dagger(\partial_\mu U)\Phi U^\dagger + U(\partial_\mu \Phi)U^\dagger - U\Phi U^\dagger(\partial_\mu U)U^\dagger \\
= U\left((U^\dagger\partial_\mu U)\Phi + \partial_\mu \Phi - \Phi(U^\dagger\partial_\mu U)\right)U^\dagger \\
= U\left(\partial_\mu \Phi + [\Phi, U^\dagger \partial_\mu U]\right)U^\dagger 
\]  

(S.3)

while the second term expands to

\[
[A'_\mu, \Phi'] = [UA_\mu U^\dagger, U\Phi U^\dagger] + [i(\partial_\mu U)U^\dagger, U\Phi U^\dagger] \\
= U\left([A_\mu, \Phi] + [iU^\dagger \partial_\mu U, \Phi]\right)U^\dagger 
\]  

(S.4)

Combining the two expansions, we arrive at

\[ D'_\mu \Phi' = U\left(\partial_\mu \Phi + [\Phi, U^\dagger \partial_\mu U] + i[A_\mu, \Phi] - [iU^\dagger \partial_\mu U, \Phi]\right)U^\dagger = U(D_\mu \Phi)U^\dagger. \]  

(S.5)

Thus, the $D_\mu \Phi(x)$ matrix transforms exactly like the $\Phi(x)$ matrix itself, which makes the $D_\mu$ derivative (4) covariant.  \textit{Q.E.D.}
Problem 1(b):
Let’s start with the second line of eq. (6). In the matrix form, the adjoint multiplet $\Phi$ is a matrix, the fundamental multiplet $\Psi$ is a column vector, and their matrix product $\Phi \Psi$ is also a column vector. The covariant derivatives acts on these matrices and vectors as

$$D_\mu \Phi = \partial_\mu \Phi + i [A_\mu, \Phi], \quad D_\mu \Psi = \partial_\mu \Psi + i A_\mu \Psi,$$  \hspace{1cm} \text{(S.6)}
while

$$D_\mu (\Phi \Psi) \overset{\text{def}}{=} \partial_\mu (\Phi \Psi) + i A_\mu (\Phi \Psi)$$
$$= (\partial_\mu \Phi) \Psi + \Phi (\partial_\mu \Psi) + i [A_\mu, \Phi] \Psi + i \Phi A_\mu \Psi \hspace{1cm} \text{(S.7)}$$

Likewise, on the third line of eq. (6), $\Xi$ is a matrix, $\Psi^\dagger$ is a row vector, and their matrix product $\Psi^\dagger \Xi$ is also a row vector. Therefore

$$D_\mu \Psi^\dagger = \partial_\mu \Psi^\dagger - i \Psi^\dagger A_\mu, \quad D_\mu \Xi = \partial_\mu \Xi + i [A_\mu, \Xi],$$  \hspace{1cm} \text{(S.8)}
while

$$D_\mu (\Psi^\dagger \Xi) \overset{\text{def}}{=} \partial_\mu (\Psi^\dagger \Xi) - i (\Psi^\dagger \Xi) A_\mu$$
$$= (\partial_\mu \Psi^\dagger) \Phi + \Psi^\dagger (\partial_\mu \Xi) - i \Psi^\dagger A_\mu \Xi + \Psi^\dagger [\Xi, A_\mu ] \hspace{1cm} \text{(S.9)}$$

Finally, on the first line of eq. (6), $\Phi$ and $\Xi$ are both $N \times N$ matrices and their product $\Phi \Xi$ is also a matrix. Consequently,

$$D_\mu (\Phi \Xi) \overset{\text{def}}{=} \partial_\mu (\Phi \Xi) + i [A_\mu, \Phi \Xi]$$
$$= (\partial_\mu \Phi) \Xi + \Phi (\partial_\mu \Xi) + i [A_\mu, \Phi] \Xi + i \Phi [A_\mu, \Xi] \hspace{1cm} \text{(S.10)}$$
$$= (D_\mu \Phi) \Xi + \Phi (D_\mu \Xi).$$

\textbf{Q.E.D.}
Problem 1(c):
For the adjoint multiplet of fields $\Phi(x)$,

$$
D_\mu D_\nu \Phi = \partial_\mu (D_\nu \Phi) + i[A_\mu, D_\nu \Phi] \\
= \partial_\mu (\partial_\nu \Phi + i[A_\nu, \Phi]) + i[A_\mu, (\partial_\nu \Phi + i[A_\nu, \Phi])] \\
= \partial_\mu \partial_\nu \Phi + i[\partial_\mu A_\nu, \Phi] + i[A_\nu, \partial_\mu \Phi] + i[A_\mu, \partial_\nu \Phi] - [A_\mu, [A_\nu, \Phi]]
$$

(S.11)

where the two blue terms are not symmetric in indices $\mu \leftrightarrow \nu$. Consequently,

$$
[D_\mu, D_\nu] \Phi = i[\partial_\mu A_\nu, \Phi] - [A_\mu, [A_\nu, \Phi]] - i[\partial_\nu A_\mu, \Phi] + [A_\nu, [A_\mu, \Phi]] \\
= i[\partial_\mu A_\nu - \partial_\nu A_\mu, \Phi] - [[A_\mu, A_\nu], \Phi]
$$

(S.12)

where the second term follows from the Jacobi identity for the matrix commutator:

$$
- [A_\mu, [A_\nu, \Phi]] + [A_\nu, [A_\mu, \Phi]] = +[A_\mu, [\Phi, A_\nu]] + [A_\nu, [A_\mu, \Phi]] \\
= -[\Phi, [A_\nu, A_\mu]] = -[[A_\mu, A_\nu], \Phi].
$$

(S.13)

Altogether, we have

$$
[D_\mu, D_\nu] \Phi = i[\{(\partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]), \Phi\}, \Phi] \equiv i[F_{\mu\nu}, \Phi] = i g F_{\mu\nu} \Phi
$$

(S.14)

where the second equality follows from the definition of the non-abelian $F_{\mu\nu}$ and the third equality from $F_{\mu\nu} = g F_{\mu\nu}$. Finally, in components

$$
ig F_{\mu\nu} \Phi = ig F^{b}_{\mu\nu} \Phi^c \times \left[ \frac{\lambda^b}{2}, \frac{\lambda^c}{2} \right] = ig F^{b}_{\mu\nu} \Phi^c \times i f^{bca} \frac{\lambda^a}{2}
$$

(S.15)

and hence $[D_\mu, D_\nu] \Phi^a = -g f^{abc} F^{b}_{\mu\nu} \Phi^c$. 

3
**Problem 1(d):**

In matrix notations, the non-abelian gauge symmetries act on vector potentials $A_\mu(x)$ according to

$$
A'_\mu(x) = U(x)A_\mu(x)U^\dagger(x) + i\partial_\mu U(x)U^\dagger(x). \quad (S.16)
$$

Taking

$$
F_{\mu\nu}(x) \overset{\text{def}}{=} \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + i [A_\mu(x), A_\nu(x)] \quad (S.17)
$$

as the definition of the tension fields $F_{\mu\nu}(x)$, we then have

$$
F'_{\mu\nu}(x) = \partial_\mu A'_\nu(x) - \partial_\nu A'_\mu(x) + i [A'_\mu(x), A'_\nu(x)], \quad (S.18)
$$

whatever that evaluates to. Specifically, the first term here evaluates to

$$
\partial_\mu A'_\nu = \partial_\mu \left( U A_\nu U^\dagger + i(\partial_\nu U)U^\dagger \right)
= U(\partial_\mu A_\nu)U^\dagger + \left[ (\partial_\mu U)U^\dagger, U A_\nu U^\dagger \right] + i(\partial_\mu \partial_\nu U)U^\dagger - i(\partial_\nu U)U^\dagger \times (\partial_\mu U)U^\dagger \quad (S.19)
$$

where the second equality follows from

$$
\partial_\mu \left( U A_\nu U^\dagger \right) = U(\partial_\nu A_\mu)U^\dagger + \left[ (\partial_\mu U)U^\dagger, U A_\nu U^\dagger \right] \quad (S.20)
$$

— cf. similar formula (S.3) — and

$$
\partial_\mu \left( (\partial_\nu U)U^\dagger \right) = (\partial_\mu \partial_\nu U)U^\dagger + (\partial_\nu U)(\partial_\mu U) = (\partial_\mu \partial_\nu U)U^\dagger - (\partial_\nu U)U^\dagger (\partial_\mu U)U^\dagger. \quad (S.21)
$$

Likewise

$$
\partial_\nu A'_\mu = U(\partial_\nu A_\mu)U^\dagger + \left[ (\partial_\nu U)U^\dagger, U A_\mu U^\dagger \right] + i(\partial_\nu \partial_\mu U)U^\dagger - i(\partial_\mu U)U^\dagger \times (\partial_\nu U)U^\dagger \quad (S.22)
$$

and hence

$$
\partial_\mu A'_\nu - \partial_\nu A'_\mu = U(\partial_\mu A_\nu - \partial_\nu A_\mu)U^\dagger + \left[ (\partial_\mu U)U^\dagger, U A_\nu U^\dagger \right] - \left[ (\partial_\nu U)U^\dagger, U A_\mu U^\dagger \right] + 0 + i \left[ (\partial_\mu U)U^\dagger, (\partial_\nu U)U^\dagger \right]. \quad (S.23)
$$
At the same time, the commutator part of the tension field transforms into

\[ i \left[ A'_\mu, A'_\nu \right] = i \left[ \left( U A_\mu U^\dagger + i(\partial_\mu U)U^\dagger \right), \left( U A_\nu U^\dagger + i(\partial_\nu U)U^\dagger \right) \right] \]

\[ = i \left[ U A_\mu U^\dagger, U A_\nu U^\dagger \right] - \left[ (\partial_\mu U)U^\dagger, U A_\nu U^\dagger \right] \]

\[ - \left[ U A_\mu U^\dagger, (\partial_\nu U)U^\dagger \right] - i \left[ (\partial_\mu U)U^\dagger, (\partial_\nu U)U^\dagger \right], \]

(S.24)

Combining eqs. (S.23) and (S.24) leads to massive cancellation of 6 out of terms on the combined right hand side. Only the first terms on right hand sides of (S.23) and (S.24) survive the cancellation, thus

\[ \partial_\mu A'_\nu - \partial_\nu A'_\mu + i \left[ A'_\mu, A'_\nu \right] = U (\partial_\mu A_\nu - \partial_\nu A_\mu)U^\dagger + i \left[ U A_\mu U^\dagger, U A_\nu U^\dagger \right] \]

\[ = U (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])U^\dagger, \]

(S.25)

or in other words,

\[ F'_{\mu\nu}(x) = U(x)F_{\mu\nu}(x)U^\dagger(x). \]

(S.26)

Q.E.D.

Problem 1(e):

There are two ways to prove the non-abelian Bianchi identity: using part (b) and the Jacobi identity for the commutators, or the hard calculation based directly on eq. (S.17).

Let me start with the easier proof. In part (b) we have proved the Leibniz rules for covariant derivatives of matrix products of adjoint and (anti)fundamental multiplets. In class — and also in part (d) we saw that the tension fields \( F_{\mu\nu}^a \) form an adjoint multiplet. So if we take any fundamental multiplet of some fields \( \Psi_i(x) \), then according to eq. (6.b)

\[ D_\lambda (F_{\mu\nu} \Psi) = (D_\lambda F_{\mu\nu}) \Psi + F_{\mu\nu}(D_\lambda \Psi). \]

(S.27)

But I we also saw in class that \( F_{\mu\nu} \Psi = -i[D_\mu, D_\nu] \Psi \), and for the same reason we should also
have $F_{\mu\nu}(D_\lambda \Psi) = -i[D_\mu, D_\nu](D_\lambda \Psi)$. Plugging these relations into eq. (S.27), we arrive at
\[
(D_\lambda F_{\mu\nu}) \times \Psi = D_\lambda (F_{\mu\nu} \Psi) - F_{\mu\nu} (D_\lambda \Psi)
\]
\[
= -iD_\lambda ([D_\mu, D_\nu] \Psi) + i[D_\mu, D_\nu] (D_\lambda \Psi) 
= -i[D_\lambda, [D_\mu, D_\nu]] \Psi.
\] (S.28)

Now let’s combine such formulae for the 3 cyclic permutations of the indices $\lambda, \mu, \nu$:
\[
(D_\lambda F_{\mu\nu}) \times \Psi = -i[D_\lambda, [D_\mu, D_\nu]] \Psi,
\]
\[
(D_\mu F_{\nu\lambda}) \times \Psi = -i[D_\mu, [D_\nu, D_\lambda]] \Psi,
\] (S.29)
\[
(D_\nu F_{\lambda\mu}) \times \Psi = -i[D_\nu, [D_\lambda, D_\mu]] \Psi,
\]
and therefore
\[
(D_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu}) \times \Psi = -i([D_\lambda, [D_\mu, D_\nu]] + [D_\mu, [D_\nu, D_\lambda]] + [D_\nu, [D_\lambda, D_\mu]]) \Psi.
\] (S.30)

The right hand side here must vanish by the Jacobi identity for the commutators, hence on the left hand side
\[
(D_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu}) \times \Psi = 0.
\] (S.31)

Moreover, this must be true for any fundamental multiplet $\Psi(x)$, which means that the matrix in () here must vanish,
\[
D_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} = 0.
\] (8)

Quod erat demonstrandum.

The second proof of the Bianchi identity follows directly from the definition (S.17) of the non-abelian tension fields and the covariant derivatives (4). Let’s spell out $D_\lambda F_{\mu\nu}$ in detail:
\[
D_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} + i[A_\lambda, F_{\mu\nu}]
\]
\[
= \partial_\lambda (\partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]) + i [A_\lambda, (\partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu])] 
\]
\[
= \partial_\lambda \partial_\mu A_\nu - \partial_\lambda \partial_\nu A_\mu + i[\partial_\lambda A_\mu, A_\nu] + i[A_\mu, \partial_\lambda A_\nu]
\]
\[
+ i[A_\lambda, \partial_\mu A_\nu] - i[A_\lambda, \partial_\nu A_\mu] - [A_\lambda, [A_\mu, A_\nu]] 
\] (S.32)
\[
= (\partial_\nu \partial_\mu A_\lambda - \partial_\lambda \partial_\nu A_\mu) + i\left([\partial_\nu A_\mu, A_\lambda] - [\partial_\mu A_\nu, A_\lambda]\right)
\]
\[
+ i\left([A_\mu, \partial_\nu A_\lambda] - [A_\lambda, \partial_\nu A_\mu]\right) - \left([A_\lambda, [A_\mu, A_\nu]]\right).
\]
On the bottom two lines here I have grouped terms in () so that after summing over cyclic permutations of the indices \( \lambda, \mu, \nu \), we get a zero sum separately for each group. Indeed,

\[
\left( \partial_{\lambda} \partial_{\mu} A_{\nu} - \partial_{\lambda} \partial_{\nu} A_{\mu} \right) + \text{cyclic} = \left( \partial_{\lambda} \partial_{\mu} A_{\nu} - \partial_{\nu} \partial_{\lambda} A_{\mu} \right) + \text{cyclic} = 0 \quad \langle \text{by inspection} \rangle,
\]

\[
\left( [\partial_\lambda A_\mu, A_\nu] - [\partial_\mu A_\nu, A_\lambda] \right) + \text{cyclic} = 0 \quad \langle \text{by inspection} \rangle,
\]

\[
\left( [A_\mu, \partial_\lambda A_\nu] - [A_\lambda, \partial_\nu A_\mu] \right) + \text{cyclic} = 0 \quad \langle \text{by inspection} \rangle, \quad \text{and}
\]

\[
[A_\lambda, [A_\mu, A_\nu]] + \text{cyclic} = 0 \quad \langle \text{by Jacobi identity} \rangle.
\]

Therefore,

\[
D_\lambda F_{\mu \nu} + \text{cyclic} \equiv D_\lambda F_{\mu \nu} + D_\mu F_{\nu \lambda} + D_\nu F_{\lambda \mu} = 0. \quad \text{(S.34)}
\]

**Problem 1(f):**

The Euler–Lagrange field equations follow from requiring zero first variation of the action \( S = \int \mathcal{L} \) under infinitesimal variation of the independent fields \( A_\mu(x) \). Let’s start by calculating the variation of the tension fields \( F_{\mu \nu} \):

\[
\delta F_{\mu \nu} \equiv \delta \left( \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + i[A_{\mu}, A_{\nu}] \right)
\]

\[
= \partial_{\mu} \delta A_{\nu} - \partial_{\nu} \delta A_{\mu} + i[\delta A_{\mu}, A_{\nu}] + i[A_{\mu}, \delta A_{\nu}]
\]

\[
= \left( \partial_{\mu} \delta A_{\nu} + i[A_{\mu}, \delta A_{\nu}] \right) - \left( \partial_{\nu} \delta A_{\mu} + i[A_{\nu}, \delta A_{\mu}] \right)
\]

\[
= D_\mu \delta A_{\nu} - D_\nu \delta A_{\mu} \quad \text{(S.35)}
\]

where we treat the matrix-valued variations \( \delta A_\nu(x) \) as adjoint fields so their covariant derivative work according to eq. (4), \( D_\mu \delta A_\nu \equiv \partial_\mu \delta A_\nu + i[A_\mu, \delta A_\nu] \) and likewise for the \( D_\nu \delta A_\mu \). In light of eq. (S.35), the trace in the Yang–Mills Lagrangian (9) varies by

\[
\delta \text{tr} \left( F_{\mu \nu} F_{\mu \nu} \right) = 2 \text{tr} \left( F_{\mu \nu} \delta F_{\mu \nu} \right) = 2 \text{tr} \left( F_{\mu \nu} (D_\mu \delta A_{\nu} - D_\nu \delta A_{\mu}) \right)
\]

\[
= 4 \text{tr} \left( F_{\mu \nu} D_\mu \delta A_{\nu} \right) \quad \langle \text{since } F_{\mu \nu} = -F_{\nu \mu} \rangle \quad \text{(S.36)}
\]

\[
= -4 \text{tr} \left( (D_\mu F_{\mu \nu}) \delta A_{\nu} \right) + 4 \partial_\mu \text{tr} \left( F_{\mu \nu} \delta A_{\nu} \right)
\]

where the last equality follows from the Leibniz rule for the two adjoint fields \( \Phi = F_{\mu \nu} \) and
\[ \Xi = \delta A_{\nu} : \]

\[
\text{tr}((D_{\mu}\Phi)\Xi) + \text{tr}(\Phi(D_{\mu}\Xi)) = \text{tr}(D_{\mu}(\Phi\Xi)) = \text{tr}(\partial_{\mu}(\Phi\Xi)) + i \text{tr}([A_{\mu}, \Phi\Xi]) = \partial_{\mu} \text{tr}(\Phi\Xi) + 0
\]  

(S.37)

(trace of a commutator is zero). Thus

\[
\delta L_{\text{YM}} = \frac{2}{g^2} \text{tr}((D_{\mu}F^{\mu\nu})\delta A_{\nu}) - \text{a total divergence}
\]  

(S.38)

so the net Yang-Mills action varies by

\[
\delta S = \frac{2}{g^2} \int d^4x \text{ tr}(D_{\mu}F^{\mu\nu}(x)\delta A_{\nu}(x)) = \frac{1}{g^2} \int d^4x \sum_a D_{\mu}F^{a\mu\nu}(x) \times \delta A_{\nu}^a(x).
\]  

(S.39)

To make this variation vanish for any infinitesimal \( \delta A_{\nu}^a(x) \) we need \( D_{\mu}F^{a\mu\nu}(x) \equiv 0 \).

---

**Problem 2(a):**

In problem 1(f) we saw that for infinitesimal variations of the gauge fields the YM Lagrangian varies by

\[
\delta L_{\text{YM}} = \frac{1}{g^2} \sum_a D_{\mu}F^{a\mu\nu} \times \delta A_{\nu}^a + \partial_{\mu}(\cdots) = \sum_a D_{\mu}F^{a\mu\nu} \times \delta A_{\nu}^a + \partial_{\mu}(\cdots).
\]  

(S.40)

Now let’s add the matter Lagrangian \( L_{\text{mat}}(\phi, D\phi) \) for some matter fields in a non-trivial multiplet (or multiplets) of the gauge symmetry. When we vary the gauge fields \( A_{\nu}^a(x) \) while keeping the matter fields \( \phi(x) \) fixed, the covariant derivatives \( D\phi \) vary due to \( igA_{\nu}^a t^a \phi \) terms in \( D_{\nu}\phi \), which leads to non-trivial variation

\[
\delta L_{\text{mat}} = \sum_a \frac{\partial L_{\text{mat}}}{\partial A_{\nu}^a} \times \delta A_{\nu}^a \equiv -\sum_a J^{a\nu} \times \delta A_{\nu}^a.
\]  

(S.41)

Altogether, the net action of the theory varies by

\[
\delta S = \int d^4x \sum_a \left( D_{\mu}F^{a\mu\nu}(x) - J^{a\nu}(x) \right) \times \delta A_{\nu}^a(x).
\]  

(S.42)

Requiring this variation to vanish for any \( \delta A_{\nu}^a(x) \) leads to the field equations

\[
D_{\mu}F^{a\mu\nu} = J^{a\nu},
\]  

(S.43)

or in matrix notations \( D_{\mu}F^{\mu\nu} = J^{\nu} \). This is the non-abelian version of the Maxwell equations
\[ \partial_\mu F^{\mu \nu} = J^\nu. \]

In the abelian EM theory, the equations \( \partial_\mu F^{\mu \nu} = J^\nu \) require the electric current to be conserved, \( \partial_\nu J^\nu = \partial_\nu \partial_\mu F^{\mu \nu} = 0 \) since \( F^{\mu \nu} = -F^{\nu \mu} \) and the derivatives commute with each other. The non-abelian tension fields \( F^{\mu \nu} \) are also antisymmetric in \( \mu \leftrightarrow \nu \), but the covariant derivatives do not commute, \( D_\mu D_\nu \neq D_\nu D_\mu \). Therefore,

\[
D_\nu J^\nu = D_\nu D_\mu F^{\mu \nu} = \frac{1}{2}[D_\mu, D_\nu]F^{\mu \nu} = \frac{i g}{2} [F^{\mu \nu}, F^{\mu \nu}] \quad \text{(S.44)}
\]

where the last equality works exactly as in problem 1(c) — the \( F^{a \mu \nu} \) fields form an adjoint multiplet of fields, and for any such multiplet packed into an hermitian \( N \times N \) matrix \( \Phi \),

\[ [D_\mu, D_\nu] \Phi = i[F^{\mu \nu}, \Phi] = ig[F^{\mu \nu}, \Phi]. \]

However, unlike a generic matrix \( \Phi \) which may commute or not commute with the \( F^{\mu \nu} \), for any \( \mu \) and \( \nu \) the \( F^{\mu \nu} \) matrix always commutes with itself. Thus,

\[ [F^{\mu \nu}, F^{\mu \nu}] = 0 \quad \text{even before summing over } \mu \text{ and } \nu. \quad \text{(S.45)} \]

Of course, after the summing over \( \mu \) and \( \nu \) we still have a zero, thus \( D_\nu D_\mu F^{\mu \nu}(x) \equiv 0 \).

Thus, consistency of the field equations (S.43) for the gauge fields requires the non-abelian currents \( J^{a \mu} \) to be covariantly conserved:

\[
D_\nu J^\nu = D_\mu D_\nu F^{\mu \nu} = 0, \quad \text{(S.46)}
\]

or in components

\[
\partial_\nu J^{a \nu} - gf^{abc} A^b_\nu J^c = 0. \quad \text{(S.47)}
\]

Note: because of the covariantizing term here, we do not have conserved net charges; alas,

\[
\frac{d}{dt} \int d^3 x \ J^{a 0}(x,t) \neq 0. \quad \text{(S.48)}
\]
Problem 2(b):
The Lagrangian for a fundamental multiplet of Dirac fermions is spelled out in eq. (13). In components,
\[ \mathcal{L}_{\text{mat}} = \overline{\Psi}(i\gamma^\mu D_\mu - m)\Psi = \overline{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi_i + \overline{\Psi}(i\gamma^\mu \times i g A^a_\mu \times (\frac{\lambda^a}{2})_i^j)\Psi_j, \]
(S.49) hence
\[ J^{a\mu} = -\frac{\partial \mathcal{L}}{\partial A^a_\mu} = g\overline{\Psi}\gamma^\mu \left(\frac{\lambda^a}{2}\right)_i^j \Psi_j \equiv g\overline{\Psi}\gamma^\mu \frac{\lambda^a}{2} \Psi. \]
(S.50)
To pack these currents into a traceless hermitian matrix \( J^\mu = \sum_a J^{a\mu} \times \frac{1}{2} \lambda^a \), we may use eq. (14), where the \( \Psi_i \) play the role of the \( \xi_i \) while the \( \overline{\Psi} j^\mu \) play the role of the \( \eta^{*j} \). Thus,
\[ (J^\mu)_i^j = \sum_a \left(J^{a\mu} = g\overline{\Psi}\gamma^\mu \frac{\lambda^a}{2} \Psi\right) \times \left(\frac{\lambda^a}{2}\right)_i^j = \frac{g}{2} \times \overline{\Psi} j^\mu \Psi_i - \frac{g}{2N} \times \overline{\Psi} k^\mu \psi_k \times \delta_i^j, \]
on in matrix notations,
\[ J^\mu = I^\mu - \frac{\text{tr}(I^\mu)}{N} \times 1 \quad \text{where} \quad \left(I^\mu\right)_i^j = \frac{g}{2} \overline{\Psi} j^\mu \Psi_i. \]
(S.51)
Note: the \( I^\mu \) matrices are hermitian but not traceless, but the \( J^\mu \) matrices are both hermitian and traceless.

Now consider the \( SU(N) \) transformation rule for these matrices. In components,
\[ \Psi_i^\dagger(x) = U_i^k(x)\Psi_k(x), \quad \overline{\Psi}^\dagger(x) = \overline{\Psi}(x)(U^\dagger(x))_\ell^j \]
(S.52) hence
\[ \left(I^\mu(x)\right)_i^j = \frac{g}{2} \overline{\Psi} j^\mu \Psi_i^\dagger(x) \]
\[ = \frac{g}{2} \overline{\Psi}^\dagger(x)(U^\dagger(x))_\ell^j \gamma^\mu U_i^k(x) \Psi_k(x) \]
\[ = U_i^k(x) \times \left(\frac{g}{2} \overline{\Psi}^\dagger(x) \gamma^\mu \Psi_k(x)\right) \times (U^\dagger(x))_\ell^j \]
\[ = U_i^k(x) \times (I^\mu(x))_k^\ell \times (U^\dagger(x))_\ell^j \]
(S.53) or in matrix notations
\[ I^\mu(x) = U(x) \times I^\mu(x) \times U^\dagger(x). \]
(S.54)
This transformation leaves the trace \( \text{tr}(I^\mu) \) invariant, while its traceless part \( J^\mu \) transforms
according to eq. (3), indeed

\[ J^\mu(x) = I^\mu(x) - \frac{\text{tr}(I^\mu(x))}{N} \times 1 \]
\[ = U(x)I^\mu(x)U^\dagger(x) - \frac{\text{tr}(U(x)I^\mu(x)U^\dagger(x)) = \text{tr}(I^\mu(x))}{N} \times 1 \]
\[ = U(x) \times \left( I^\mu(x) - \frac{\text{tr}I^\mu(x)}{N} \times 1 \right) \times U^\dagger(x) \]
\[ = U(x) \times J^\mu(x) \times U^\dagger(x). \] (S.55)

In other words, the currents \( J^{a\mu}(x) \) transform into each other as members of an adjoint multiplet, \textit{quod erat demonstrandum}.

\textbf{Problem 2(c):}
First, let \( \xi(x) \) be a fundamental multiplet of some fields while \( \eta^\dagger(x) \) is an antifundamental multiplet, and let’s prove the Leibniz rule for the adjoint multiplet of the form \( Q^a = \eta^\dagger \lambda^a \xi' \):

\[ D_\mu(\eta^\dagger \lambda^a \xi) = (D_\mu \eta^\dagger) \lambda^a \Psi + \eta^\dagger \lambda^a (D_\mu \xi). \] (S.56)

Proof:

\[ D_\mu(\eta^\dagger \lambda^a \xi) \equiv \partial_\mu(\eta^\dagger \lambda^a \xi) - g f^{abc} A^b_\mu (\eta^\dagger \lambda^c \xi) \]
\[ = (\partial_\mu \eta^\dagger) \lambda^a \xi + \eta^\dagger \lambda^a (\partial_\mu \xi) - g A^b_\mu \eta^\dagger (f^{abc} \lambda^c = -\frac{i}{2} [\lambda^a, \lambda^b]) \xi \]
\[ = \left( \partial_\mu \eta^\dagger - \frac{ig}{2} A^b_\mu \eta^\dagger \lambda^b \right) \lambda^a \xi + \eta^\dagger \lambda^a \left( \partial_\mu \xi + \frac{ig}{2} A^b_\mu \lambda^b \xi \right) \] (S.57)
\[ = (D_\mu \eta^\dagger) \lambda^a \xi + \eta^\dagger \lambda^a (D_\mu \xi). \]

Now that we have this general formula, let’s apply it to the non-abelian currents \( J^{a\mu} = (g/2) \overline{\Psi} \gamma^\mu \lambda^a \Psi \). Using \( \Psi \) for \( \xi \) and \( \overline{\Psi} \gamma^\mu \) for \( \eta \), we arrive at

\[ D_\mu J^{a\mu} = (g/2)(D_\mu \overline{\Psi}) \gamma^\mu \lambda^a \Psi + (g/2)\overline{\Psi} \lambda^a \gamma^\mu (D_\mu \Psi). \] (S.58)

Note: in this formula, the order of the \( \lambda^a \) and \( \gamma^\mu \) matrices is unimportant since they act on unrelated indices of the Dirac fields.
Now let’s apply the Euler-Lagrange equations for the fermionic fields, which are simply the covariant Dirac equation and its conjugate,

\[ i\gamma^\mu D_\mu \Psi = m\Psi, \quad -iD_\mu \overline{\Psi}\gamma^\mu = m\overline{\Psi}. \] (S.59)

Plugging these equations of motion into eq. (S.58), we immediately obtain

\[
D_\mu J^{a\mu} = \left(\frac{g}{2}\right)\left(D_\mu \overline{\Psi}\gamma^\mu = im\overline{\Psi}\right) \lambda^a \Psi + \left(\frac{g}{2}\right)\overline{\Psi}\lambda^a \left(\gamma^\mu D_\mu \Psi = -im\Psi\right)
= im\left(\frac{g}{2}\right) \times \overline{\Psi}\lambda^a \Psi - im\left(\frac{g}{2}\right) \times \Psi\lambda^a \Psi
= 0.
\] (S.60)

*Quod erat demonstrandum.*

**Problem 3:**

Regardless of the nature of the matter field multiplet \((m)\) from the symmetry group’s point of view — it could be a vector, a tensor, a spinor, whatever — let me treat the \(\Psi_\alpha(x)\) as components of some big column vector \(\Psi(x)\) of length size\((m)\) so I can use matrix notations for the \(\left(T^a_{(m)}\right)\) without bothering with indices \(\alpha, \beta, \ldots\). Thus instead of \(\left(T^a_{(m)}\right)_\alpha^\beta \Psi_\beta(x)\) I will write simply \(T^a_{(m)} \Psi(x)\).

A local symmetry parametrized by infinitesimal \(\Lambda^a(x)\) acts on the gauge and matter fields according to

\[
\delta A^a_\mu(x) = -\partial_\mu \Lambda^a(x) - f^{abc} \Lambda^b(x) A^c_\mu(x), \quad \delta \Psi(x) = i\Lambda^a(x) T^a_{(m)} \Psi(x). \quad (15 + 16)
\]

Consequently, the covariant derivatives

\[
D_\mu \Psi(x) = \partial_\mu \Psi(x) + iA^a_\mu(x) T^a_{(m)} \Psi(x).
\] (17)
change by

\[
\delta(D_\mu \Psi(x)) = D_\mu \delta \Psi(x) + (\delta D_\mu \text{ due to } \delta A_\mu) \Psi(x) \\
= \partial_\mu \delta \Psi(x) + i A_\mu(x)^a T^a_{(m)} \delta \Psi(x) + i \delta A_\mu(x)^a T^a_{(m)} \Psi(x) \\
= i \partial_\mu \Lambda^a(x) T^a_{(m)} \Psi(x) + i \Lambda^a(x) T^a_{(m)} \partial_\mu \Psi(x) \\
- A_\mu^b T^b_{(m)} \Lambda^a(x) T^a_{(m)} \Psi(x) \\
- i \partial_\mu \Lambda^a(x) T^a_{(m)} \Psi(x) - i f^{abc} A_\mu^b(x) A_\mu^c(x) T^a_{(m)} \Psi(x) \\
+ i f^{cab} \Lambda^a(x) A_\mu^b(x) T^c_{(m)} \Psi(x) \\
= i \Lambda^a(x) T^a_{(m)} \partial_\mu \Psi(x) - \Lambda^a(x) A_\mu^b(x) (T^b_{(m)} T^a_{(m)} + i f^{cab} T^c_{(m)}) \Psi(x) \\
= i \Lambda^a(x) T^a_{(m)} D_\mu \Psi(x).
\]  

(S.61)

Now let’s simplify the matrix inside the big () in the second term on the bottom line. The matrices \(T^a_{(m)}\) represent the Lie algebra of the local symmetry, so they obey the same commutation relations as the generators \(\hat{T}^a\),

\[
[T^a_{(m)}, T^b_{(m)}] = i f^{abc} T^c_{(m)}. \tag{S.62}
\]

Also, \(f^{abc}\) is totally antisymmetric so \(f^{cab} = + f^{abc}\). Consequently

\[
i f^{cab} T^c_{(m)} = f^{abc} T^c_{(m)} = T^a_{(m)} T^b_{(m)} - T^b_{(m)} T^a_{(m)} \implies T^a_{(m)} T^b_{(m)} + i f^{cab} T^c_{(m)} = T^a_{(m)} T^b_{(m)}, \tag{S.63}
\]

and therefore

\[
\delta(D_\mu \Psi(x)) = i \Lambda^a(x) T^a_{(m)} \partial_\mu \Psi(x) - \Lambda^a(x) A_\mu^b(x) T^a_{(m)} T^b_{(m)} \Psi(x) \\
= i \Lambda^a(x) T^a_{(m)} \left( \partial_\mu \Psi(x) + i A_\mu^b(x) T^b_{(m)} \Psi(x) \right) \\
= i \Lambda^a(x) T^a_{(m)} D_\mu \Psi(x).
\]  

(S.64)

Thus the \(D_\mu \Psi(x)\) transforms under the infinitesimal local symmetries exactly like the \(\Psi(x)\) itself, which makes the derivative \(D_\mu\) covariant. \(\Box\)
Problem 3(*):
Under finite gauge symmetries \( \mathcal{G}(x) \in G \), the field multiplet \( \Psi(x) \) in representation \((m)\) transform to

\[
\Psi(x) \rightarrow R^{(m)}(\mathcal{G}(x)) \Psi(x)
\]  

(S.65)

— or in components

\[
\Psi_\alpha(x) \rightarrow \left( R^{(m)}(\mathcal{G}(x)) \right)^\beta_\alpha \Psi_\beta(x)
\]  

(S.66)

— where \( R^{(m)}(\mathcal{G}) \) is the unitary matrix representing group element \( \mathcal{G} \) in multiplet \((m)\); for \( \mathcal{G} = \exp(-i\Theta^a T^a) \) for some finite real parameters \( \Theta^a \),

\[
\left( R^{(m)}(\mathcal{G}(x)) \right)^\beta_\alpha = \exp(-i\Theta^a T^a_{(m)})^\beta_\alpha.
\]  

(S.67)

To simplify the form of covariant derivatives \( D_\mu \Psi(x) \), let me combine the gauge fields \( A_{\mu}^a(x) \) with matrices \( T^a_{(m)} \) representing the Lie algebra of \( G \) in multiplet \((m)\) into a size\((m)\) \( \times \) size\((m)\) matrix-valued connection

\[
A_{\mu}^{(m)}(x) \overset{\text{def}}{=} \sum_a A_{\mu}^a(x) \times \left( T^a_{(m)} \right) \implies D_\mu \Psi(x) = \partial_\mu \Phi(x) + i A_{\mu}^{(m)}(x) \Psi(x).
\]  

(S.68)

To make sure these derivatives transform covariantly under finite local symmetries (S.65), the matrix-valued connection \( A_{\mu}^{(m)}(x) \) should transform as

\[
A_{\mu}^{(m)}(x) \rightarrow R^{(m)}(\mathcal{G}(x)) \times A_{\mu}^{(m)}(x) \times \left( R^{(m)}(\mathcal{G}(x)) \right)^{-1} + i \partial_\mu R^{(m)}(\mathcal{G}(x)) \times \left( R^{(m)}(\mathcal{G}(x)) \right)^{-1}.
\]  

(S.69)

This works similarly to the \( SU(N) \) symmetry I have discussed in class:

\[
D_\mu \Psi \rightarrow D'_\mu \Psi' = \partial_\mu \left( R^{(m)} \Psi \right) + i \left( R^{(m)} A_{\mu} \right) \left( R^{(m)} \right)^{-1} + i \partial_\mu R^{(m)} \times \left( R^{(m)} \right)^{-1} \times R^{(m)} \Psi
\]

\[
= \partial_\mu R^{(m)} \times \Psi + R^{(m)} \times \partial_\mu \Psi + i R^{(m)} A_{\mu} \times \Psi - \partial_\mu R^{(m)} \times \Psi
\]

\[
= R^{(m)} \times D_\mu \Psi.
\]  

(S.70)

But the real problem here is to make sure that transforms (S.69) are consistent with having the same gauge fields \( A_{\mu}^a(x) \) for all multiplets of the gauge group.
Before I write down the transformation law for the $\mathcal{A}_{\mu}^a(x)$ fields in a multiplet-independent manner, let me note that the symmetries $\mathcal{G}(x)$ should be continuous functions of $x$. Consequently, for an infinitesimal displacement $\epsilon^\mu$, $\mathcal{G}(x + \epsilon) \times \mathcal{G}^{-1}(x) = 1 + O(\epsilon)$. But for any Lie group member infinitesimally close to unity, its displacement from unity is a linear combination of the Lie algebra generators $\hat{T}^a$, thus

$$\mathcal{G}(x + \epsilon) \times \mathcal{G}^{-1}(x) = 1 + i\epsilon^\mu C_\mu^a(x) \hat{T}^a + O(\epsilon^2) \quad \text{(S.71)}$$

for some real coefficients $C_\mu^a(x)$. In terms of derivatives of $\mathcal{G}(x)$,

$$\partial_\mu \mathcal{G}(x) \times \mathcal{G}^{-1}(x) = iC_\mu^a(x) \hat{T}^a. \quad \text{(S.72)}$$

Given the coefficients $C_\mu^a(x)$ in this equation, I can write an explicit formula for the finite gauge transform of the non-abelian gauge fields $\mathcal{A}_{\mu}^a(x)$:

$$\mathcal{A}_{\mu}^a(x) = R^{ab}_{\text{adj}}(\mathcal{G}(x)) \times \mathcal{A}_{\mu}^b(x) - C_\mu^a(x) \quad \text{(S.73)}$$

where $R^{ab}_{\text{adj}}(\mathcal{G})$ represents $\mathcal{G}$ in the adjoint multiplet of the Lie group $G$.

Now let me show that the gauge transform (S.73) leads to eqs. (S.69) for all multiplets $(m)$ of the Lie group $G$. Any representation of $G$ must respect the group product,

$$R^{(m)}(\mathcal{G}_2 \times \mathcal{G}_1) = R^{(m)}(\mathcal{G}_2) \times R^{(m)}(\mathcal{G}_1). \quad \text{(S.74)}$$

Also, in the infinitesimal neighborhood of the unity,

$$R^{(m)}(1 + i\epsilon^a \hat{T}^a) = 1 + i\epsilon^a T^a_{(m)}, \quad \text{(S.75)}$$

Consequently, for any multiplet $(m)$,

$$R^{(m)}(\mathcal{G}(x + \epsilon)) \times \left(R^{(m)}(\mathcal{G}(x))\right)^{-1} = R^{(m)}(\mathcal{G}(x + \epsilon) \times \mathcal{G}^{-1}(x))$$

$$= R^{(m)}(1 + i\epsilon^\mu C_\mu^a \hat{T}^a)$$

$$= 1 + i\epsilon^\mu C_\mu^a T^a_{(m)} \quad \text{(S.76)}$$

and hence

$$\partial_\mu R^{(m)}(\mathcal{G}(x)) \times \left(R^{(m)}(\mathcal{G}(x))\right)^{-1} = iC_\mu^a T^a_{(m)} \quad \text{(S.77)}$$

with exactly the same coefficients $C_\mu^a(x)$ as in eq. (S.72). Therefore, the second term in
eq. (S.69) for the transformation of the $A^{(m)}_{\mu}(x) = A^a_{\mu}(x) T^a_{(m)}$ agrees with the $-C^a_{\mu}(x)$ term in eq. (S.73) for the transformation of the component fields $A^a_{\mu}(x)$.

As to the first term in eq. (S.69), it agrees with the first term in eq. (S.73) thanks to the Lemma (18): for any multiplet $(m)$ and any group element $G \in G$,

$$ R^{(m)}(G) \times T^b_{(m)} \times \left( R^{(m)}(G) \right)^{-1} = \sum_a T^a_{(m)} R^a_{\text{adj}}(G), \quad (S.78) $$

hence

$$ R^{(m)}(G) \times A^b_{\mu} T^a_{(m)} \times \left( R^{(m)}(G) \right)^{-1} = A^b_{\mu} T^a_{(m)} R^a_{\text{adj}}(G) = T^a_{(m)} \left( R^a_{\text{adj}}(G) A^b_{\mu} \right). \quad (S.79) $$

This completes my proof that the gauge fields $A^a_{\mu}(x)$ transforming under finite local symmetries according to eq. (S.73) makes the derivatives $D_{\mu}$ covariant for all multiplets of the symmetry group $G$.

To complete these solutions, let me also prove the Lemma (S.78).

I assume $G = \exp(-i\Theta^a \hat{T}^a)$ for some real parameters $\Theta^a$ and hence $R^{(m)}(G) = \exp(-i\Theta^a T^a_{(m)})$.

By the multiple-commutator formula

$$ e^A Be^{-A} = B + [A, B] + \frac{1}{2}[[A, [A, B]] + \frac{1}{6}[A, [A, [A, B]]] + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} [A, \cdots [A, B] \cdots]^{n \text{times}} \quad (S.80) $$

we have

$$ R^{(m)}(G) T^a_{(m)} \left( R^{(m)}(G) \right)^{-1} = \exp \left( -i\Theta^b T^b_{(m)} \right) T^a_{(m)} \exp \left( +i\Theta^b T^b_{(m)} \right) $$

$$ = T^a_{(m)} - i \left[ \Theta^b T^b_{(m)} , T^a_{(m)} \right] + \frac{(-i)^2}{2} \left[ \Theta^c T^c_{(m)} , \Theta^b T^b_{(m)} , T^a_{(m)} \right] $$

$$ + \frac{(-i)^3}{3!} \left[ \Theta^d T^d_{(m)} , \Theta^c T^c_{(m)} , \Theta^b T^b_{(m)} , T^a_{(m)} \right] + \cdots \quad (S.81) $$

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where all the commutators follow from the Lie algebra:

\[-i \left[ \Theta^b T^b_{(m)}, T^a_{(m)} \right] = \Theta^b f^{bac} T^c_{(m)},
\]

\[(-i)^2 \left[ \Theta^c T^c_{(m)}, \left[ \Theta^b T^b_{(m)}, T^a_{(m)} \right] \right] = \Theta^b f^{bad} \times (-i) \left[ \Theta^c T^c_{(m)}, T^d_{(m)} \right] = \left( \Theta^b f^{bad} \right) \left( \Theta^c f^{cde} \right) T^e_{(m)}, \tag{S.82}\]

\[(-i)^3 \left[ \Theta^d T^d_{(m)}, \left[ \Theta^c T^c_{(m)}, \left[ \Theta^b T^b_{(m)}, T^a_{(m)} \right] \right] \right] = \left( \Theta^b f^{bae} \right) \left( \Theta^c f^{cef} \right) \left( \Theta^d f^{dfg} \right) T^g_{(m)}, \tag{S.83}\]

All the contractions of \( \Theta \)'s with structure constants on the right hand sides here may be interpreted in terms of the adjoint multiplet of the group \( G \) where \((T^a_{\text{adj}})^{bc} = -i f^{abc}\):

\[\Theta^b f^{bac} = \left( i \Theta^b T^b_{\text{adj}} \right)^{ac}, \]

\[\left( \Theta^b f^{bad} \right) \left( \Theta^c f^{cde} \right) = \left( i \Theta^b T^b_{\text{adj}} \right)^{ad} \left( i \Theta^c T^c_{\text{adj}} \right)^{de} = \left( \left( i \Theta^b T^b_{\text{adj}} \right)^2 \right)^{ae}, \tag{S.83}\]

\[\left( \Theta^b f^{bae} \right) \left( \Theta^c f^{cef} \right) \left( \Theta^d f^{dfg} \right) = \left( \left( i \Theta^b T^b_{\text{adj}} \right)^3 \right)^{ag}.\]

Combining eqs. (S.81) through (S.83), we obtain

\[R^{(m)}(G) T^a_{(m)} \left( R^{(m)}(G) \right)^{-1} = T^a_{(m)} + \left( i \Theta^b T^b_{\text{adj}} \right)^{ac} T^c_{(m)} + \frac{1}{2} \left( \left( i \Theta^b T^b_{\text{adj}} \right)^2 \right)^{ac} T^e_{(m)} + \frac{1}{6} \left( \left( i \Theta^b T^b_{\text{adj}} \right)^3 \right)^{ag} T^g_{(m)} + \cdots \]

\[= \left( \exp \left( i \Theta^b T^b_{\text{adj}} \right) \right)^{ac} T^c_{(m)} \]

\[\langle \text{using antisymmetry of } (T^b_{\text{adj}})^{ac} = -(T^b_{\text{adj}})^{ca} \rangle \]

\[= T^c_{(m)} \left( \exp \left( -i \Theta^b T^b_{\text{adj}} \right) \right)^{ca} \]

\[= T^c_{(m)} \times R^c_{\text{adj}}(G), \tag{S.84}\]

which proves the Lemma (S.78).
Problem 4(a):
When a gauge symmetry is spontaneously broken, the gauge fields acquire masses — which come from the gauge-covariant kinetic terms for the scalar fields with non-zero VEVs (vacuum expectation values). The simplest way to separate the vector mass terms from shifted scalars’ kinetic energies and scalar-vector interactions is to freeze the scalar fields to their VEVs. Indeed, let’s freeze \( \Phi(x) \equiv \langle \Phi \rangle = C \times 1_{N \times N} \). Then according to eq. (21),

\[
D_\mu \langle \Phi \rangle = ig'B_\mu \langle \Phi \rangle + igL_\mu \langle \Phi \rangle - ig\langle \Phi \rangle R_\mu
\]

\[
= ig'C B_\mu \times 1_{N \times N} + igC \times (L_\mu - R_\mu)
\]

\[
= ig'C B_\mu \times 1_{N \times N} + \frac{igC}{2}(L_\mu^a - R_\mu^a) \times \lambda^a,
\]

and consequently

\[
\text{tr}\left((D_\mu \langle \Phi \rangle^\dagger)(D^\mu \langle \Phi \rangle)\right) = Ng'^2C^2 \times B_\mu B^\mu + \frac{g^2C^2}{2} \sum_a (L_\mu^a - R_\mu^a)(L^a_\mu - R^a_\mu).
\]

Thus, the abelian \( B_\mu \) field has mass \( M_B^2 = 2Ng'^2C^2 \) while the non-abelian fields \( L_\mu^a \) and \( R_\mu^a \) have non-diagonal mass terms. To diagonalize those terms, let’s mix the fields according to

\[
V_\mu^a = \frac{1}{\sqrt{2}} (L_\mu^a + R_\mu^a), \quad X_\mu^a = \frac{1}{\sqrt{2}} (L_\mu^a - R_\mu^a),
\]

where the \( 1/\sqrt{2} \) coefficients make the \( V_\mu^a \) and \( X_\mu^a \) canonically normalized, i.e.

\[
\mathcal{L}_{\text{kin}}^{L,R} = -\frac{1}{4} \sum_a \left( \left( \partial_{[\mu} L_{\nu]}^a \right)^2 + \left( \partial_{[\mu} R_{\nu]}^a \right)^2 \right) = -\frac{1}{4} \sum_a \left( \left( \partial_{[\mu} X_{\nu]}^a \right)^2 + \left( \partial_{[\mu} V_{\nu]}^a \right)^2 \right).
\]

In terms of the \( V_\mu^a \) and \( X_\mu^a \), the mass terms for \( L_\mu^a \) and \( R_\mu^a \) in eq. (86) become

\[
\mathcal{L}_{\mu}^{\text{masses}} = g^2C^2 \times X_\mu^a X^{a\mu}.
\]

Thus, the \( V_\mu^a \) fields remain massless while the \( X_\mu^a \) acquire common mass \( M_X^2 = 2g^2C^2 \).
Problem 4(b):
To write down an effective theory for the massless fields, we simply freeze all the massive fields \( B_{\mu}, X_{\mu}^a, \tilde{\varphi}_1, \) and \( \xi_1 \) (in the notations of the homework set #11). only the massless \( V^a_{\mu} \) remain un-frozen. In other words, we let

\[
\Phi(x) \equiv \langle \Phi \rangle = C \times 1_{N \times N}, \quad B_{\mu}(x) \equiv 0, \quad L_{\mu}^a(x) = R_{\mu}^a(x) = \frac{1}{\sqrt{2}} V^a_{\mu}(x), \quad \text{(S.90)}
\]

and then substitute these values into the Lagrangian (4.9). According to eq. (S.85), for fields as in eq. (S.90) \( D_{\mu} \Phi = 0 \), so the only un-frozen terms in the Lagrangian are

\[
\mathcal{L}^{\text{unfrozen}} = -\frac{1}{2} \text{tr}(L_{\mu\nu} L^{\mu\nu}) - \frac{1}{2} \text{tr}(R_{\mu\nu} R^{\mu\nu}) \quad \langle \text{for } L_{\mu\nu}^a = R_{\mu\nu}^a \rangle
\]

\[
= -\text{tr}(L_{\mu\nu} L^{\mu\nu}) = -\frac{1}{2} \sum_a (L_{\mu\nu}^a)^2
\]

\[
= -\frac{1}{4} \sum_a (V^a_{\mu\nu})^2 \quad \text{(S.91)}
\]

— which is precisely the Yang-Mills Lagrangian for the canonically normalized tension fields

\[
V^a_{\mu\nu} = \frac{L^a_{\mu\nu} + R^a_{\mu\nu}}{\sqrt{2}} \rightarrow \sqrt{2} L^a_{\mu\nu} \quad \text{when } L^a_{\mu\nu} = R^a_{\mu\nu}. \quad \text{(S.92)}
\]

of the un-broken \( SU(N)_V \) gauge theory. Indeed, in terms of the canonically normalized \( SU(N)_V \) potential fields \( V^a_{\mu} \),

\[
V^a_{\mu\nu} = \sqrt{2} \left( \partial_{\mu} L^a_{\nu} - \partial_{\nu} L^a_{\mu} - gf^{abc} L^b_{\mu} L^c_{\nu} \right)
\]

\[
= \sqrt{2} \left( \partial_{\mu} V^a_{\nu} - \partial_{\nu} V^a_{\mu} - gf^{abc} \frac{V^b_{\mu} V^c_{\nu}}{\sqrt{2}} \right) \quad \text{(S.93)}
\]

\[
= \partial_{\mu} V^a_{\nu} - \partial_{\nu} V^a_{\mu} - \frac{g}{\sqrt{2}} f^{abc} V^b_{\mu} V^c_{\nu}.
\]

The coefficient of the non-abelian last term on the bottom line is the gauge coupling of the unbroken \( SU(N)_V \) gauge group

\[
g_v = \frac{g}{\sqrt{2}}. \quad \text{(S.94)}
\]
Problem 4(*):
For $g_L \neq g_R$, the covariant derivatives of the scalar fields become

$$D_\mu \Phi = \partial_\mu \Phi + ig'B_\mu \Phi + ig_LL_\mu \Phi - ig_R \Phi R_\mu.$$  \hfill (S.95)

As in part (a), the mass terms for the vector fields obtain from plugging $\langle \Phi \rangle$ into these covariant derivatives and then expanding the kinetic terms for the scalars:

$$D^{-\mu} \langle \Phi \rangle = ig'C B_\mu \times 1_{N \times N} + iC(g_LL_\mu - g_R R_\mu) \times \frac{\lambda^a}{2}$$ \hfill (S.96)

and hence

$$\mathcal{L} \supset \text{tr}(D_\mu \Phi^\dagger D^\mu \Phi) \supset Ng'^2C^2 \times B_\mu B^\mu + \frac{C^2}{2} \times (gL L_\mu^a - gR R_\mu^a)(gL L^{a\mu} - gR R^{a\mu}).$$ \hfill (S.97)

As in part (a), the abelian gauge fields gets mass $M_B^2 = 2Ng'^2C^2$, while the non-abelian vector mass is more tricky.

Let’s define the coupling $\tilde{g}$ and the mixing angle $\theta$ according to

$$g_L = \tilde{g} \times \cos \theta, \quad g_R = \tilde{g} \times \sin \theta \quad \Rightarrow \quad \tilde{g}^2 = g_L^2 + g_R^2, \quad \tan \theta = \frac{g_R}{g_L}. \hfill (S.98)$$

Then the non-abelian mass term in eq. (S.97) becomes

$$\frac{C^2g'^2}{2} \times (L_\mu^a \cos \theta - R_\mu^a \sin \theta)^2,$$ \hfill (S.99)

which tells us which particular combination of the non-abelian gauge fields become massive.

Indeed, if we let

$$X_\mu^a = \cos \theta \times L_\mu^a - \sin \theta \times R_\mu^a,$$
$$Y_\mu^a = \sin \theta \times L_\mu^a + \cos \theta \times R_\mu^a,$$ \hfill (S.100)

then both combinations of vector fields are canonically normalized — indeed,

$$\mathcal{L}_{L,R}^{\text{kin}} = -\frac{1}{4} \sum_a \left( \left( \partial_{[\mu} L_{\nu]}^a \right)^2 + \left( \partial_{[\mu} R_{\nu]}^a \right)^2 \right) = -\frac{1}{4} \sum_a \left( \left( \partial_{[\mu} X_{\nu]}^a \right)^2 + \left( \partial_{[\mu} Y_{\nu]}^a \right)^2 \right), \hfill (S.101)$$

— while the mass term (S.99) becomes

$$\mathcal{L}_{L,R}^{\text{mass}} = \frac{C^2g'^2}{2} \times X_\mu^a X^{a\mu}.$$ \hfill (S.102)

Thus, the $X_\mu^a$ fields have mass $M_X = \tilde{g}C$, while the $Y_\mu^a$ fields remain massless.
Now let’s derive the effective Lagrangian for just the massless vector fields \( Y_\mu^a(x) \) while freezing all the other fields, \( i.e. \) setting \( \Phi(x) \equiv \langle \Phi \rangle, B_\mu(x) \equiv 0, \) and \( X_\mu^a(x) \equiv 0. \) In terms of the \( L_\mu^a \) and \( R_\mu^a \) fields, this means
\[
L_\mu^a = Y_\mu^a \times \sin \theta, \quad R_\mu^a = Y_\mu^a \times \cos \theta, \quad (S.103)
\]
or in terms of the group-normalized gauge fields
\[
L_\mu = g_L \times L_\mu = g_L \sin \theta \times Y_\mu, \quad R_\mu = g_R \times R_\mu = g_R \cos \theta \times Y_\mu. \quad (S.104)
\]
However, for the mixing angle \( \theta \) related to the couplings as in eq. (S.98), we have
\[
g_L \sin \theta = g_R \cos \theta = \frac{g_L g_R}{g} \overset{\text{def}}{=} g_Y, \quad (S.105)
\]
and therefore
\[
L_\mu(x) = R_\mu(x) = g_Y \times Y_\mu(x) \overset{\text{def}}{=} \mathcal{Y}_\mu(x). \quad (S.106)
\]
In terms of the \( \mathcal{Y}_\mu(x) \) gauge field, the non-abelian tension fields are
\[
\mathcal{L}_{\mu\nu}(x) = \mathcal{R}_{\mu\nu}(x) = \mathcal{Y}_{\mu\nu}(x) = \partial_\mu \mathcal{Y}_\nu(x) - \partial_\nu \mathcal{Y}_\mu(x) + i[\mathcal{Y}_\mu, \mathcal{Y}_\nu] \quad (S.107)
\]
— precisely as for a group-normalized \( SU(N) \) connection \( \mathcal{Y}_\mu(x) \) — while the net YM Lagrangian is
\[
\mathcal{L}_{YM} = -\frac{1}{2g_Y^2} \text{tr}(\mathcal{L}_{\mu\nu} \mathcal{L}^{\mu\nu}) - \frac{1}{2g^2} \text{tr}(\mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu}) = -\frac{1}{2} \left( \frac{1}{g_L^2} + \frac{1}{g_R^2} \right) \times \text{tr}(\mathcal{Y}_{\mu\nu} \mathcal{Y}^{\mu\nu}) \quad (S.108)
\]
— precisely as for the \( SU(N) \) YM theory with inverse gauge coupling
\[
\frac{1}{g_Y^2} = \frac{1}{g_L^2} + \frac{1}{g_R^2}. \quad (S.109)
\]
After a bit of algebra, this formula becomes
\[
g_Y = \frac{g_L g_R}{\sqrt{g_L^2 + g_R^2}} , \quad (24)
\]
in perfect agreement with \( \mathcal{Y}_\mu^a(x) = g_Y \times Y_\mu^a(x) \) for the canonically normalized gauge fields \( Y_\mu^a(x). \)