Phase Space Factors

For quantum transitions to un-bound states — for example, an atom emitting a photon, or a radioactive decay, or scattering — which is a kind of unbound $\rightarrow$ unbound transition, - the transition rate is given by the Fermi’s golden rule:

$$\Gamma \overset{\text{def}}{=} \frac{d\text{probability}}{dt} = \frac{2\pi\rho}{\hbar} \times \left| \langle\text{final}|\hat{T}|\text{initial}\rangle \right|^2$$  \hspace{1cm} (1)

where $\hat{T} = \hat{H}_{\text{perturbation}} + \text{higher order corrections}$, and $\rho$ is the density of final states,

$$\rho = \frac{dN_{\text{final states}}}{dE_{\text{final}}}.$$  \hspace{1cm} (2)

For an example, consider an atom emitting a photon of specific polarization $\lambda$. Using the large-box normalization for the photon’s states, we have

$$dN_{\text{final}} = \left(\frac{L}{2\pi}\right)^3 d^3k_{\gamma} = \frac{L^3}{(2\pi)^3} \times k_{\gamma}^2 dk_{\gamma} d^2\Omega_{\gamma}$$  \hspace{1cm} (3)

while $dE_{\text{final}} = dE_{\gamma} = \hbar c \times dk_{\gamma}$, hence

$$\rho = L^3 \times \frac{k_{\gamma}^2}{(2\pi)^3\hbar c} \times d^2\Omega_{\gamma}.$$  \hspace{1cm} (4)

The $L^3$ factor here cancels against the $L^{-3/2}$ factor in the matrix element $\langle\text{atom'} + \gamma|\hat{T}|\text{atom}\rangle$ due to the photon’s wave function in the large-box normalization. As to the remaining $d^2\Omega_{\gamma}$ factor, we should integrate over it to get the total decay rate, or divide by it to get the partial emission rate $d\Gamma/d\Omega$ for the photons emitted in a particular direction, thus

$$\frac{d\Gamma(\lambda)}{d\Omega} = \frac{k^2}{(2\pi)^3\hbar c} \times L^3 \left| \langle\text{atom'} + \gamma(k,\lambda)|\hat{T}|\text{atom}\rangle \right|^2,$$

$$\Gamma_{\text{total}} = \int d\Omega \sum_{\lambda} \frac{k^2}{(2\pi)^3\hbar c} \times L^3 \left| \langle\text{atom'} + \gamma(k,\lambda)|\hat{T}|\text{atom}\rangle \right|^2.$$  \hspace{1cm} (5)

In relativistic normalization of quantum states and matrix elements, there are no $L^{-3/2}$ factors but instead there are $\sqrt{2E}$ factors for each final-state or initial state particle, and
they must be compensated by dividing the density of states $\rho$ by the $\prod_i (2E_i)$. Also, we must allow for motion of all the final-state particles (i.e., both the photon and the recoiled atom) but impose the momentum conservation as a constraint. Thus, for a decay of 1 initial particle into $n$ final particles,

$$\Gamma = \frac{1}{2E_{\text{in}}} \int \frac{d^3p'_1}{(2\pi)^3 2E_1'} \cdots \int \frac{d^3p'_n}{(2\pi)^3 2E_n'} \left| \langle p'_1, \ldots, p'_n | \mathcal{M} | p_{\text{in}} \rangle \right|^2 \times (2\pi^4)\delta^{(4)}(p'_1 + \cdots + p'_n - p_{\text{in}}),$$

(6)

where the $\delta$ function takes care of both momentum conservation and of the denominator $dE$ in the density-of-states factor (2). Likewise, the transition rate for a generic $2 \rightarrow n$ scattering process is given by

$$\Gamma = \frac{1}{2E_1 \times 2E_2} \int \frac{d^3p_1'}{(2\pi)^3 2E_1'} \cdots \int \frac{d^3p_n'}{(2\pi)^3 2E_n'} \left| \langle p'_1, \ldots, p'_n | \mathcal{M} | p_1, p_2 \rangle \right|^2 \times (2\pi^4)\delta^{(4)}(p'_1 + \cdots + p'_n - p_1 - p_2).$$

(7)

In terms of the scattering cross-section $\sigma$, the rate (7) is $\Gamma = \sigma \times$ flux of initial particles. In the large-box normalization the flux is $L^{-3}|\mathbf{v}_1 - \mathbf{v}_2|$, so in the continuum normalization it's simply the relative speed $|\mathbf{v}_1 - \mathbf{v}_2|$. Consequently, the total scattering cross-section is given by

$$\sigma_{\text{tot}} = \frac{1}{4E_1 E_2 |\mathbf{v}_1 - \mathbf{v}_2|} \int \frac{d^3p'_1}{(2\pi)^3 2E_1'} \cdots \int \frac{d^3p_n'}{(2\pi)^3 2E_n'} \left| \langle p'_1, \ldots, p'_n | \mathcal{M} | p_1, p_2 \rangle \right|^2 \times (2\pi^4)\delta^{(4)}(p'_1 + \cdots + p'_n - p_1 - p_2).$$

(8)

In particle physics, all the factors in eqs (6) or (8) besides the matrix elements — as well as the integrals over such factors — are collectively called the phase space factors.

A note on Lorentz invariance of decay rates or cross-sections. The matrix elements $\langle \text{final} | \mathcal{M} | \text{initial} \rangle$ are Lorentz invariant, and so are all the integrals over the final-particles’ momenta and the $\delta$-functions. The only non-invariant factor in the decay-rate formula (6) is the pre-integral $1/E_{\text{in}}$, hence the decay rate of a moving particle is

$$\Gamma(\text{moving}) = \Gamma(\text{rest frame}) \times \frac{M}{E}$$

(9)

where $M/E$ is precisely the time dilation factor in the moving frame.
As to the scattering cross-section, it should be invariant under Lorentz boosts along the initial axis of scattering, thus the same cross-section in any frame where \( p_1 \parallel p_2 \). This includes the \textit{lab frame} where one of the two particles is initially at rest, the \textit{center-of-mass frame} where \( p_1 + p_2 = 0 \), and any other frame where the two particles collide head-on. And indeed, the pre-integral factor in eq. (8) for the cross-section

\[
\frac{1}{4E_1E_2|\mathbf{v}_1 - \mathbf{v}_2|} = \frac{1}{4|E_1\mathbf{p}_2 - E_2\mathbf{p}_1|}
\]

is invariant under Lorentz boosts along the scattering axis.

Let’s simplify eq. (8) for a 2 particle \( \rightarrow 2 \) particle scattering process in the center-of-mass frame where \( p_1 + p_2 = 0 \). In this frame, the pre-exponential factor (10) becomes

\[
\frac{1}{4|\mathbf{p}|} \times (E_1 + E_2)
\]

while the remaining phase space factors amount to

\[
P_{\text{int}} = \int \frac{d^3p_1'}{(2\pi)^3} 2E_1' \int \frac{d^3p_2'}{(2\pi)^3} 2E_2' \frac{(2\pi)^4 \delta(3)(p_1' + p_2') \delta(E_1' + E_2' - E_{\text{net}})}{16\pi^2 E_1'E_2'} \times \delta(E_1' + E_2' - E_{\text{tot}})
\]

\[
= \int \frac{d^3p_1'}{(2\pi)^3} \times 2E_1' \times 2E_2' \frac{(2\pi) \delta(E_1'(p_1') + E_2'(-p_1') - E_{\text{net}})}{16\pi^2 E_1'E_2'}
\]

\[
= \int d^2\Omega_{\mathbf{p}'} \times \int_0^{\infty} dp' \frac{p'^2}{16\pi^2 E_1'E_2'} \times \delta(E_1' + E_2' - E_{\text{tot}})
\]

\[
= \int d^2\Omega_{\mathbf{p}'} \left[ \frac{p'^2}{16\pi^2 E_1'E_2'} \right] \frac{d(E_1' + E_2')}{dp'} \bigg|_{E_1' + E_2' = E_{\text{tot}}}
\]

On the last 3 lines here \( E_1' = E_1'(p_1') = \sqrt{p'^2 + m_1'^2} \) while \( E_2' = E_2'(p_2' = -p_1') = \sqrt{p'^2 + m_2'^2} \). Consequently,

\[
\frac{dE_1'}{dp'} = \frac{p'}{E_1'}, \quad \frac{dE_2'}{dp'} = \frac{p'}{E_2'}
\]

hence

\[
\frac{d(E_1' + E_2')}{dp'} = \frac{p'}{E_1'} + \frac{p'}{E_2'} = \frac{p'}{E_1'E_2'} \times (E_2' + E_1' = E_{\text{tot}}),
\]
and therefore
\[ P_{\text{int}} = \frac{1}{16\pi^2} \times \frac{p'}{E_{\text{tot}}} \times \int d^2\Omega_{p'}. \]  
(15)

Including the pre-integral factor (11), we arrive at the net phase space factor
\[ P = \frac{p'}{p} \times \frac{1}{64\pi^2 E_{\text{tot}}^2} \times \int d^2\Omega_{p'}. \]  
(16)

The matrix element \( M \) for the scattering should be put inside the direction-angle integral in this phase-space formula. Thus, the total scattering cross-section is
\[ \sigma_{\text{tot}}(1 + 2 \rightarrow 1' + 2') = \frac{p'}{p} \times \frac{1}{64\pi^2 E_{\text{cm}}^2} \times \int d^2\Omega \ |\langle p_1' + p_2' | M | p_1 + p_2\rangle|^2, \]  
(17)

while the partial cross-section for scattering in a particular direction is
\[ \frac{d\sigma(1 + 2 \rightarrow 1' + 2')}{d\Omega_{\text{cm}}} = \frac{p'}{p} \times \frac{1}{64\pi^2 E_{\text{cm}}^2} \times |\langle p_1' + p_2' | M | p_1 + p_2\rangle|^2. \]  
(18)

Note: the total cross-section is the same in frames where the initial momenta are collinear, but in the partial cross-section, \( d\Omega \) depends on the frame of reference, so eq. (18) applies only in the center-of-mass frame. Also, the \( E_{\text{cm}} \) factor in denominators of both formulæ stands for the net energy in the center-of-mass frame. In frame-independent terms,
\[ E_{\text{cm}}^2 = (p_1 + p_2)^2 = (p_1' + p_2')^2 = s. \]  
(19)

Finally, let me write down the phase-space factor for a 2-body decay (1 particle \( \rightarrow \) 2 particles) in the rest frame of the initial particle. The under-the-integral factors for such a decay are the same as in eq. (15) for a \( 2 \rightarrow 2 \) scattering, but the pre-integral factor is \( 1/2M_{\text{init}} \) instead of the (11), thus
\[ P = \frac{p'}{32\pi^2 M^2}, \]  
(20)

meaning
\[ \frac{d\Gamma(0 \rightarrow 1' + 2')}{d\Omega} = \frac{p'}{32\pi^2 M^2} \times |\langle p_1' + p_2' | M | p_0\rangle|^2, \]  
(21)
\[ \Gamma(0 \rightarrow 1' + 2') = \frac{p'}{32\pi^2 M^2} \times \int d^2\Omega \ |\langle p_1' + p_2' | M | p_0\rangle|^2. \]  
(22)