ELECTRIC DIPOLES

In these notes, I write down the electric field of a dipole, and also the net force and the torque on a dipole in the electric field of other charges. For simplicity, I focus on ideal dipoles — also called pure dipoles — where the distance $a$ between the positive and the negative charges is infinitesimal, but the charges are so large that the dipole moment $p$ is finite.

Electric Field of a Dipole

The potential due to an ideal electric dipole $p$ is

$$V(r) = \frac{p \cdot \hat{r}}{4\pi\varepsilon_0 r^2},$$

or in terms of spherical coordinates where the North pole ($\theta = 0$) points in the direction of the dipole moment $p$,

$$V(r, \theta) = \frac{p}{4\pi\varepsilon_0} \frac{\cos \theta}{r^2}.$$  

Taking (minus) gradient of this potential, we obtain the dipole’s electric field

$$\mathbf{E} = \frac{p}{4\pi\varepsilon_0} \left( \frac{2 \cos \theta}{r^3} \nabla r + \frac{\sin \theta}{r^2} \nabla \theta \right) = \frac{p}{4\pi\varepsilon_0} \frac{1}{r^3} \left( 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right).$$

In this formula, the unit vectors $\hat{r}$ and $\hat{\theta}$ themselves depend on $\theta$ and $\phi$. Translating them to Cartesian unit vectors, we have

$$\hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z},$$
$$\hat{\theta} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z},$$

hence

$$2 \cos \theta \hat{r} + \sin \theta \hat{\theta} = 3 \sin \theta \cos \theta (\cos \phi \hat{x} + \sin \phi \hat{y}) + (2 \cos^2 \theta - \sin^2 \theta = 3 \cos^2 \theta - 1) \hat{z},$$

(5)
and therefore
\[ E_x(r, \theta, \phi) = \frac{p}{4\pi\epsilon_0} \frac{3\sin \theta \cos \theta \cos \phi}{r^3}, \]
\[ E_y(r, \theta, \phi) = \frac{p}{4\pi\epsilon_0} \frac{3\sin \theta \cos \theta \sin \phi}{r^3}, \]
\[ E_z(r, \theta, \phi) = \frac{p}{4\pi\epsilon_0} \frac{3\cos^2 \theta - 1}{r^3}. \]

In terms of the \((x, y, z)\) coordinates
\[ E_x(x, y, z) = \frac{p}{4\pi\epsilon_0} \frac{3xz}{(x^2 + y^2 + z^2)^{5/2}}, \]
\[ E_y(x, y, z) = \frac{p}{4\pi\epsilon_0} \frac{3yz}{(x^2 + y^2 + z^2)^{5/2}}, \]
\[ E_z(x, y, z) = \frac{p}{4\pi\epsilon_0} \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}}, \]

or in vector notations,
\[ \mathbf{E}(\mathbf{r}) = \frac{3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{4\pi\epsilon_0 r^3}. \]

Here is the picture of the dipole’s electric field lines (in the \(xz\) plane):
Force and Torque on a Dipole

Now consider an ideal dipole $\mathbf{p}$ placed in an electric field $\mathbf{E}(x, y, z)$ due to some other sources. If this electric field is uniform, there is no net force on the dipole but there is a net torque. Indeed, the force $F_+ = +q\mathbf{E}$ acting on the positive charge cancels the opposite force $F_- = -q\mathbf{E} = -F_+$ acting on the negative charge — so the net force is zero — but the two forces are acting at different points, which causes a torque. Specifically, the net torque of the two forces is

$$\mathbf{\tau} = \mathbf{r}_+ \times F_+ + \mathbf{r}_- \times F_- = (\mathbf{r}_+ - \mathbf{r}_-) \times q\mathbf{E} = q(\mathbf{r}_+ - \mathbf{r}_-) \times \mathbf{E}, \quad (8)$$

or in terms of the dipole moment $\mathbf{p} = q(\mathbf{r}_+ - \mathbf{r}_-)$,

$$\mathbf{\tau} = \mathbf{p} \times \mathbf{E}. \quad (9)$$

This torque vanishes when the dipole moment $\mathbf{p}$ is parallel to the electric field $\mathbf{E}$. Otherwise, the torque twists the dipole trying to make it align with the field, $\mathbf{p} \rightarrow \mathbf{p}' \parallel \mathbf{E}$.

When the electric field $\mathbf{E}(x, y, z)$ is not uniform, the two charges of the dipole feel slightly different electric fields, so the net force on the dipole does not quite vanish:

$$\mathbf{F}^{\text{net}} = q(\mathbf{E}(\mathbf{r}_+) - \mathbf{E}(\mathbf{r}_-)) \neq 0. \quad (10)$$

but for small displacements $\mathbf{a} = \mathbf{r}_+ - \mathbf{r}_-$ between the charges, we may expand the difference between the electric fields acting on them into a power series in $\mathbf{a}$. Let $\mathbf{r}_\pm = \mathbf{r}_m \pm \frac{1}{2} \mathbf{a}$ where $\mathbf{r}_m$ is the middle of the dipole; then

$$\mathbf{E}(\mathbf{r}_\pm) = \mathbf{E}(\mathbf{r}_m) \pm \left(\frac{1}{2} \mathbf{a} \cdot \nabla\right) \mathbf{E}\bigg|_{\mathbf{r}_m} + \frac{1}{2} \left(\frac{1}{2} \mathbf{a} \cdot \nabla\right)^2 \mathbf{E}\bigg|_{\mathbf{r}_m} + \frac{1}{6} \left(\frac{1}{2} \mathbf{a} \cdot \nabla\right)^3 \mathbf{E}\bigg|_{\mathbf{r}_m} + \cdots, \quad (11)$$

and hence the difference

$$\mathbf{E}(\mathbf{r}_+) - \mathbf{E}(\mathbf{r}_-) = (\mathbf{a} \cdot \nabla) \mathbf{E}\bigg|_{\mathbf{r}_m} + \frac{1}{24} (\mathbf{a} \cdot \nabla)^3 \mathbf{E}\bigg|_{\mathbf{r}_m} + \cdots. \quad (12)$$

Consequently, the net force on the dipole is

$$\mathbf{F}^{\text{net}} = q(\mathbf{a} \cdot \nabla) \mathbf{E}\bigg|_{\mathbf{r}_m} + \frac{q}{24} (\mathbf{a} \cdot \nabla)^3 \mathbf{E}\bigg|_{\mathbf{r}_m} + \cdots. \quad (13)$$

For a physical dipole with a finite distance $a$ between the two charges, we must generally
take into account all the subleading terms in this expansion. But for an ideal dipole we take
the limit $a \to 0$ while $q \times p$ stays finite, so for any $n > 1$ $q \times a^n \to 0$. This makes all the
subleading terms in eq. (13) negligible compared to the leading term, therefore the net force
on an ideal dipole is simply

$$F_{\text{net}} = (p \cdot \nabla)E \bigg|_{r_m}. \quad (14)$$

The force (14) is conservative and stems from the potential energy

$$U(r_m, \hat{p}) = -p \cdot E(r_m). \quad (15)$$

Indeed, (minus) the gradient of this $U$ WRT the dipole’s location $r_m$ taken for a fixed dipole
orientation $\hat{p}$ produces the force (14),

$$-\nabla U \bigg|_{\text{fixed} \hat{p}} = (p \cdot \nabla)E \bigg|_{r_m} = F_{\text{net}}. \quad (16)$$

Also, variation of the potential energy (15) under infinitesimal rotations of the dipole mo-
moment $p$ accounts for the torque

$$\vec{\tau} = p \times E. \quad (9)$$

To be precise, this is the torque relative to the dipole center $r_m$. In a non-uniform electric
field, the torque relative to some other point $r_0$ has an extra term due to the net force (14)
on the dipole, thus

$$\vec{\tau}_{\text{net}} = (r_m - r_0) \times F_{\text{net}} + p \times E(r_m) = (r_m - r_0) \times (p \cdot \nabla)E \bigg|_{r_m} + p \times E(r_m). \quad (17)$$

This net torque may also be obtained from the potential energy $U$ — or rather its infinitesimal
variation under simultaneous rotations of the dipole moment vector $p$ and of the displacement
$r_m - r_0$ of the dipole from the reference point — but I am not going to work it out in these
notes.