Mandelstam Variables

Consider any kind of a $2 \text{ particles} \to 2 \text{ particles}$ process

\[
\begin{array}{c}
1' \\
\downarrow \\
1 \\
\uparrow \\
2 \\
\uparrow \\
2'
\end{array}
\] (1)

The 4-momenta $p_1^\mu$, $p_2^\mu$, $p_1'^\mu$, and $p_2'^\mu$ of the 2 incoming and 2 outgoing particles satisfy 8 constraints: the on-shell conditions for each particle

\[
p_1^2 = m_1^2, \quad p_2^2 = m_2^2, \quad p_1'^2 = m_1'^2, \quad p_2'^2 = m_2'^2,
\] (2)

and the net 4-momentum conservation

\[
p_1^\mu + p_2^\mu = p_1'^\mu + p_2'^\mu.
\] (3)

Altogether, this gives us $4 \times 4 - 8 = 8$ independent momentum variables, and the number of independent Lorentz-invariant combinations of these variables is only $8 - 6 = 2$.

However, for practical purposes it’s is often convenient to use 3 Lorentz-invariant variables with a fixed sum,

\[
s = (p_1 + p_2)^2 = (p_1' + p_2')^2, \\
t = (p_1 - p_1')^2 = (p_2' - p_2)^2, \\
u = (p_1 - p_2')^2 = (p_1' - p_2)^2.
\] (4)

Indeed,

\[
s + t + u = (p_1 + p_2)^2 + (p_1 - p_1')^2 + (p_1 - p_2')^2
\]

\[
= 3p_1^2 + p_2^2 + p_1'^2 + p_2'^2 + 2(p_1p_2) - 2(p_1p_1') - 2(p_1p_2')
\]

\[
= p_1^2 + p_2^2 + p_1'^2 + p_2'^2 + 2p_1 \times (p_1 + p_2 - p_1' - p_2' = 0)
\]

\[
= p_1^2 + p_2^2 + p_1'^2 + p_2'^2
\]

\[
= m_1^2 + m_2^2 + m_1'^2 + m_2'^2.
\] (5)

The $s$, $t$, and $u$ are called Mandelstam variables after Stanley Mandelstam who introduced them back in 1958.
In the center-of-mass frame where \( p_1 + p_2 = p_{1\prime} + p_{2\prime} = 0 \), \( \sqrt{s} \) is the total energy of the colliding particles, \( \sqrt{s} = E_1 + E_2 = E_{1\prime} + E_{2\prime} \). Also, for an elastic collision in the CM frame, \( t \) parametrizes the scattering angle according to 
\[
(t) = -(p_{1\prime} - p_1)^2 = -p^2 \times (1 - \cos \theta).
\]
Hence, the Lorentz-invariant definitions (4) translate the CM-frame energy and the CM-frame scattering angle to any other frame of reference.

All Lorentz-invariant combinations of the four momenta \( p^\mu_1, p^\mu_2, p'^\mu_1, \) and \( p'^\mu_2 \) can be expressed in terms of the Mandelstam variables. For example, the Lorentz products \( k^\mu k'^\mu \) of any two momenta are

\[
\begin{align*}
2(p_1 p_2) &= (p_1 + p_2)^2 - p_1^2 - p_2^2 = s - m_1^2 - m_2^2, \\
2(p_1' p_2') &= (p_1' + p_2')^2 - p_1'^2 - p_2'^2 = s - m_1'^2 - m_2'^2, \\
2(p_1 p_1') &= p_1^2 + p_1'^2 - (p_1 - p_1')^2 = m_1^2 + m_1'^2 - t, \\
2(p_2 p_1') &= p_2^2 + p_1'^2 - (p_2 - p_1')^2 = m_2^2 + m_1'^2 - t, \\
2(p_1 p_2') &= p_1^2 + p_2'^2 - (p_1 - p_2')^2 = m_1^2 + m_2'^2 - u, \\
2(p_2 p_1') &= p_2^2 + p_1'^2 - (p_2 - p_1')^2 = m_2^2 + m_1'^2 - u.
\end{align*}
\]

In particular, for an elastic scattering of 2 same-mass particles

\[
\begin{align*}
2(p_1 p_2) &= 2(p_1' p_2') = s - 2m^2, \\
2(p_1 p_1') &= 2(p_2 p_2') = 2m^2 - t, \\
2(p_1 p_2') &= 2(p_2 p_1') = 2m^2 - u.
\end{align*}
\]

For future reference, let me give you similar formulae for the \( e^- e^+ \rightarrow \mu^- \mu^+ \) pair-production,

\[
\begin{align*}
2(p_1 p_2) &= s - 2m_e^2 \approx s, \\
2(p_1 p_1') &= s - 2M_\mu^2, \\
2(p_2 p_1') &= 2(p_1' p_2') = M_\mu^2 + m_e^2 - t \approx M_\mu^2 - t, \\
2(p_1 p_2') &= 2(p_2 p_1') = M_\mu^2 + m_e^2 - u \approx M_\mu^2 - u.
\end{align*}
\]
and for the $e^- e^+ \to \gamma \gamma$ annihilation process $p_- + p_+ \to k_1 + k_2$,

\[
\begin{align*}
  s + t + u &= 2m_e^2, \\
  2(p_- p_+) &= s - 2m_e^2, \\
  2(k_1 k_2) &= s, \\
  2(p_- k_1) &= 2(p_+ k_2) = m_e^2 - t, \\
  2(p_- k_2) &= 2(p_+ k_1) = m_e^2 - u.
\end{align*}
\]