QED Feynman Rules in the Counterterm Perturbation Theory

The simplest version of QED (Quantum ElectroDynamics) has only 2 field types — the electromagnetic field $A^\mu$ and the electron field $\Psi$ — and its physical Lagrangian is

$$L_{\text{phys}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\Psi}(i\gamma^\mu D_\mu - m_e)\Psi = -\frac{1}{4}F_{\mu\nu}^2 + \overline{\Psi}(i\not\partial - m)\Psi + eA_\mu\overline{\Psi}\gamma^\mu\Psi. \quad (1)$$

The bare Lagrangian of the perturbation theory has a similar form, except for the bare coupling $e_{\text{bare}}$ instead of the physical coupling $e$, the bare electron mass $m_{\text{bare}}$ instead of the physical mass $m$, and the bare fields $A^\mu_{\text{bare}}(x)$ and $\Psi_{\text{bare}}(x)$ instead of the renormalized fields $A^\mu(x)$ and $\Psi(x)$. By convention, the fields strength factors $Z$ for the EM and the electron fields are called respectively the $Z_3$ and the $Z_2$, while the $Z_1$ is the electric charge renormalization factor. Thus,

$$A^\mu_{\text{bare}}(x) = \sqrt{Z_3} \times A^\mu(x), \quad \Psi_{\text{bare}}(x) = \sqrt{Z_2} \times \Psi(x), \quad (2)$$

and plugging these bare fields into the bare Lagrangian we obtain

$$L_{\text{bare}} = -\frac{Z_3}{4}F_{\mu\nu}F^{\mu\nu} + Z_2\overline{\Psi}(i\not\partial - m_{\text{bare}})\Psi + Z_1 e \times A_\mu\overline{\Psi}\gamma^\mu\Psi \quad (3)$$

where

$$Z_1 \times e = Z_2 \sqrt{Z_3} \times e_{\text{bare}} \quad (4)$$

by definition of the $Z_1$.

As usual in the counterterm perturbation theory, we split

$$L_{\text{bare}} = L_{\text{phys}} + L^{\text{counter terms}}_{\text{terms}} \quad (5)$$

where the physical Lagrangian $L_{\text{phys}}$ is exactly as in eq. (1) while the counterterms comprise the difference. Specifically,

$$L^{\text{counter terms}}_{\text{terms}} = -\frac{\delta_3}{4} \times F_{\mu\nu}F^{\mu\nu} + \delta_2 \times \overline{\Psi}(i\not\partial)\Psi - \delta_m \times \overline{\Psi}\Psi + e\delta_1 \times A_\mu\overline{\Psi}\gamma^\mu\Psi \quad (6)$$

for

$$\delta_3 = Z_3 - 1, \quad \delta_2 = Z_2 - 1, \quad \delta_1 = Z_1 - 1, \quad \delta_m = Z_2 m_{\text{bare}} - m_{\text{phys}}. \quad (7)$$

Actually, the bare Lagrangian (5) is not the whole story, since in the quantum theory
the EM field $A^\mu(x)$ needs to be gauge-fixed. In the Feynman gauge, or in similar Lorentz-invariant gauges, the gauge fixing amounts to adding an extra gauge-symmetry breaking term to the Lagrangian,

$$\mathcal{L}_{\text{bare}} = \mathcal{L}_{\text{phys}} + \mathcal{L}_{\text{gauge fixing}} + \mathcal{L}_{\text{counter terms}}$$

for

$$\mathcal{L}_{\text{gauge fixing}} = -\frac{1}{2\xi} (\partial_\mu A^\mu)^2$$

where $\xi$ is a constant parametrizing a specific gauge. In the Feynman gauge $\xi = 1$.

In the counterterm perturbation theory, we take the free Lagrangian to be the quadratic part of the physical Lagrangian plus the gauge fixing term, thus

$$\mathcal{L}_{\text{free}} = \overline{\Psi}(i\not\!\partial - m)\Psi - \frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2\xi} (\partial_\mu A^\mu)^2$$

(10)

(\text{where } m \text{ is the physical mass of the electron}), while all the other terms in the bare Lagrangian — the physical coupling $eA_\mu \overline{\Psi} \gamma^\mu \Psi$ and all the counterterms (6) — are treated as perturbations. Consequently, the QED Feynman rules have the following propagators and vertices:

- The electron propagator

$$\alpha \begin{array}{c} \rightarrow \\ p \end{array} \beta = \left[ i \not\!\partial - m + i0 \right]_{\alpha\beta} = \frac{i(p + m)_{\alpha\beta}}{p^2 - m^2 + i0}$$

(11)

where $\alpha$ and $\beta$ are the Dirac indices, usually not written down.

- The photon propagator

$$\begin{array}{c} \mu \\ k \end{array} \begin{array}{c} \rightarrow \\ k \end{array} \begin{array}{c} \nu \\ k \end{array} = \frac{-i}{k^2 + i0} \times \left( g^{\mu\nu} + (\xi - 1) \frac{k^\mu k^\nu}{k^2 + i0} \right).$$

(12)

In the Feynman gauge $\xi = 1$ this propagator simplifies to

$$\begin{array}{c} \mu \\ k \end{array} \begin{array}{c} \rightarrow \\ k \end{array} \begin{array}{c} \nu \\ k \end{array} = \frac{-ig^{\mu\nu}}{k^2 + i0}.$$
• The physical vertex

\[ \alpha \quad \mu \quad \beta \]

\[ \left( +ie\gamma^\mu \right)_{\alpha\beta}. \]  \hspace{1cm} (14)

The Dirac indices \(\alpha\) and \(\beta\) of the fermionic lines are usually not written down.

🌟 And then there are three kinds of the counterterm vertices:

\[ \alpha \quad \mu \quad \beta \]

\[ +ie\delta_1 \times (\gamma^\mu)_{\alpha\beta}, \]  \hspace{1cm} (15)

\[ \alpha \quad \beta \]

\[ +i(\delta_2 \times p - \delta_3)_{\alpha\beta}, \]  \hspace{1cm} (16)

\[ \mu \quad \nu \]

\[ -i\delta_3 \times (g^{\mu\nu} k^2 - k^\mu k^\nu). \]  \hspace{1cm} (17)