Finite Multiplets of the Spin(3,1) Group.

In these notes I classify all the finite multiplets of the continuous Lorentz group \(SO^+(3,1)\) or rather of its double-covering group Spin(3,1). The notes are interspersed with optional exercises for the students. The solutions to the exercises will appear in a separate page. I presume you read these notes after finishing your homework#5 and homework#6, so you should be familiar with the Lorentz \(\hat{J}\) and \(\hat{K}\) generators and their Dirac spinor representations. In these notes, it’s convenient to re-organize the \(\hat{J}\) and \(\hat{K}\) generators into two non-hermitian 3-vectors

\[
\hat{J}^+ = \frac{1}{2}(\hat{J} + i\hat{K}) \quad \text{and} \quad \hat{J}^- = \frac{1}{2}(\hat{J} - i\hat{K}) = \hat{J}^\dagger. \tag{1}
\]

1. Show that the two 3-vectors commute with each other, \([\hat{J}^k_+, \hat{J}^\ell_-] = 0\), while the components of each 3-vector satisfy angular momentum commutation relations, \([\hat{J}^k_+, \hat{J}^\ell_+] = i\epsilon^{k\ell m} \hat{J}^m_+\) and \([\hat{J}^k_-, \hat{J}^\ell_-] = i\epsilon^{k\ell m} \hat{J}^m_-\).

By themselves, the 3 \(\hat{J}^k_+\) generate a symmetry group similar to rotations of a 3D space, but since the \(\hat{J}^k_+\) are non-hermitian, the finite irreducible multiplets of this symmetry are non-unitary analytic continuations (to complex “angles”) of the ordinary angular momentum multiplets \((j)\) of spin \(j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\). Likewise, the finite irreducible multiplets of the symmetry group generated by the \(\hat{J}^k_-\) are analytic continuations of the spin-\(j\) multiplets of angular momentum. Moreover, the two symmetry groups commute with each other, so the finite irreducible multiplets of the net Lorentz symmetry are tensor products \((j_+) \otimes (j_-)\) of the \(\hat{J}^+\) and \(\hat{J}^-\) multiplets. In other words, distinct finite irreducible multiplets of the Lorentz symmetry may be labeled by two integer or half-integer ‘spins’ \(j_+\) and \(j_-\), while the states within such a multiplet are \(|j_+, j_-, m_+, m_-\rangle\) for \(m_+ = -j_+, \ldots, +j_+\) and \(m_- = -j_-, \ldots, +j_-\).

The simplest non-trivial Lorentz multiplets are two inequivalent doublets, the left-handed Weyl spinor \(2\) and the right-handed Weyl spinor \(2^*\). The \(2\) multiplet has \(j_+ = \frac{1}{2}\) while \(j_- = 0\), hence \(\hat{J}^+_+\) acts as \(\frac{1}{2}\sigma\) while \(\hat{J}^-_-\) does not act at all, or in terms of the \(\hat{J}\) and \(\hat{K}\) generators...
\[ J = \frac{1}{2} \sigma \] while \[ K = -\frac{i}{2} \sigma \]. The conjugate \( 2^* \) multiplet has \( j_- = \frac{1}{2} \) while \( j_+ = 0 \), hence \( \hat{J} \) acts as \( \frac{1}{2} \sigma \) while \( \hat{K} \) acts as \( +\frac{i}{2} \sigma \).

2. Check that these two doublets are indeed the LH Weyl spinors and the RH Weyl spinor from the [homework set #10] (problem 2).

3. Check that for finite Lorentz symmetries, the \( 2 \times 2 \) matrices \( M_L \) and \( M_R \) representing them in the LH and the RH Weyl spinor multiplets have determinant = 1.

The complex (but not necessary unitary) \( 2 \times 2 \) matrices of unit determinant form a non-compact group called the \( SL(2, \mathbb{C}) \). This group is isomorphic to the \( Spin(3,1) \), the double cover of the continuous Lorentz group \( SO^+(3,1) \). Just like the \( SU(2) \) is isomorphic to the \( Spin(3) \), the double cover of the \( SO(3) \) rotation group.

For the \( Spin(3) = SU(2) \) group, one can construct a multiplet of any spin \( j \) from a symmetric tensor product of \( 2j \) doublets. This procedure gives us an object \( \Phi_{\alpha_1, \ldots, \alpha_{2j}} \) with \( 2j \) spinor indices \( \alpha_1, \ldots, \alpha_{2j} = 1, 2 \) that’s totally symmetric under permutation of those indices and transforms under an \( SU(2) \) symmetry \( U \) as

\[
\Phi_{\alpha_1, \alpha_2, \ldots, \alpha_{2j}} \to U^\alpha_{\alpha_1} U^\beta_{\alpha_2} \cdots U^\beta_{\alpha_{2j}} \Phi_{\beta_1, \beta_2, \ldots, \beta_{2j}}. \tag{2}
\]

For integer \( j \), such objects are equivalent to tensors of the \( SO(3) \); for example, for \( j = 2 \) \( \Phi_{\alpha\beta} \equiv \Phi_{\beta\alpha} \) is equivalent to an \( SO(3) \) vector \( \vec{\Phi} \).

For the Lorentz group \( Spin(3,1) \) we have a similar situation — any multiplet can be constructed by tensoring together a bunch of two-component spinors of the \( SL(2, \mathbb{C}) \). But unlike the \( SU(2) \), the \( SL(2, \mathbb{C}) \) has two different spinors \( 2 \not\cong 2^* \) transforming under different rules. Notationally, we shall distinguish them by different index types: the un-dotted Greek indices belong to spinor that transform according to \( M \in SL(2, \mathbb{C}) \) while the dotted Greek indices belong to spinors that transform according to \( M^* \):

\[
(\psi_L)_{\alpha} \to M^\alpha_{\beta}(\psi_L)_\beta \not\cong (\sigma_2 \psi_R)_\gamma \to M^\gamma_{\delta}(\sigma_2 \psi_R)_\delta, \quad M \in SL(2, \mathbb{C}). \tag{3}
\]

Combining such spinors to make a multiplet with ‘spins’ \( j_+ \) and \( j_- \), we make an object \( \Phi_{\alpha_1, \ldots, \alpha_{(2j+)}, \gamma_1, \ldots, \gamma_{(2j-)}} \) with \( 2j_+ \) un-dotted indices and \( 2j_- \) dotted indices. \( \Phi_- \) is totally
symmetric under permutations of the un-dotted indices with each other or dotted indices with each other, but there is no symmetry between dotted and un-dotted indices. Under an $SL(2, \mathbb{C})$ symmetry $M$, the un-dotted indices transform according to $M$ while the dotted indices transform according to the $M^*$, thus

$$
\Phi_{\alpha_1 \ldots \alpha_{(2j+)}; \gamma_1 \ldots \gamma_{(2j-)}} \rightarrow M^{\beta_1}_{\alpha_1} \ldots M^{\beta_{(2j+)}}_{\alpha_{(2j+)}} \times M^*_{\gamma_1} \ldots M^*_{\gamma_{(2j-)}} \times \Phi_{\beta_1 \ldots \beta_{(2j+)}; \delta_1 \ldots \delta_{(2j-)}}.
$$

(4)

Of particular importance among such multi-spinors is the bi-spinor $V_{\alpha \dot{\gamma}}$ with $j_+ = j_- = \frac{1}{2}$ — it is equivalent to the Lorentz vector $V^\mu$. The map between bi-spinors and Lorentz vectors involves four hermitian $2 \times 2$ matrices $\sigma_\mu = (1, \sigma)$. In $SL(2, \mathbb{C})$ terms, each $\sigma_\mu$ matrix has one dotted and one un-dotted index, thus $(\sigma_\mu)_{\alpha \dot{\gamma}}$. Using the $\sigma_\mu$, we may re-cast any Lorentz vector $V^\mu$ as a matrix

$$
V^\mu \rightarrow V^\mu \sigma_\mu = V^0 + V \cdot \sigma
$$

(5)

an hence as a $(\frac{1}{2}, \frac{1}{2})$ bi-spinor

$$
V_{\alpha \dot{\gamma}} = (V^\mu \sigma_\mu)_{\alpha \dot{\gamma}} = V^0 \delta_{\alpha \dot{\gamma}} + V \cdot \sigma_{\alpha \dot{\gamma}}.
$$

(6)

Under an $SL(2, \mathbb{C})$ symmetry, the bi-spinor transforms as

$$
V_{\alpha \dot{\gamma}} \rightarrow V'_{\alpha \dot{\gamma}} = M^{\beta}_{\alpha} M^*_\delta \, V'_{\beta \delta},
$$

(7)

or in matrix form,

$$
V^\mu \sigma_\mu \rightarrow V'^\mu \sigma_\mu = M \,(V^\mu \sigma_\mu) \, M^\dagger.
$$

(8)

Since the four matrices $\sigma_\mu$ form a complete basis of $2 \times 2$ matrices, eq. (8) defines a linear transform $V'^\mu = L'_\nu (M) V^\nu$.

4. Prove that for any $SL(2, \mathbb{C})$ matrix $M$, the transform $L'_\nu (M)$ defined by eq. (8) is real (real $V'^\mu$ for real $V^\mu$), Lorentzian (preserves $V'_\mu V'^\nu = V_\mu V^\nu$) and orthochronous.

Hint: prove and use $\det(V_\mu \sigma^\mu) = V_\mu V^\mu$.

* For extra challenge, show that this transform is proper, $\det(L) = +1$. 

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5. Verify that this $SL(2, \mathbb{C}) \to SO^+(3,1)$ map respects the group law, $L(M_2 M_1) = L(M_2)L(M_1)$.

6. Show that for the $L(M)$ defined by eq. (8), the LH Weyl spinor representation of $L(M)$ is $M_L(L) = M$ while the RH Weyl spinor representation is $\bar{M} = \sigma_2 M^* \sigma_2$.

In general, any $(j_+, j_-)$ multiplet of the $SL(2, \mathbb{C})$ with integer net spin $j_+ + j_-$ is equivalent to some kind of a Lorentz tensor. (Here, we include the scalar and the vector among the tensors.) For example, the $(1, 1)$ multiplet is equivalent to a symmetric, traceless 2–index tensor $T^\mu\nu = -T^\nu\mu$, $T_\mu^\mu = 0$. For $j_+ \neq j_-$ the representation is complex, but one can make a real tensor by combining two multiplets with opposite $j_+$ and $j_-$, for example the $(1, 0)$ and the $(0, 1)$ multiplets are together equivalent to the antisymmetric 2–index tensor $F^\mu\nu = -F^\nu\mu$.

7. Verify the above examples.

   Hint: For any kind of angular momentum, $(j = \frac{1}{2}) \otimes (j = \frac{1}{2}) = (j = 1) \oplus (j = 0)$. 
