This whole exam concerns a QFT comprising the following fields: A charged Dirac spinor $\Psi_C(x)$ (charge = $-e$), a neutral Dirac spinor $\Psi_N(x)$, a charged scalar $\Phi(x)$ (charge = $-e$), and the electromagnetic field $A^\mu(x)$. The physical Lagrangian of the theory is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \Phi^* D^\mu \Phi - M_C^2 \Phi^* \Phi + \overline{\Psi}_C (i\gamma^\mu - M_C) \Psi_C + \overline{\Psi}_N (i\gamma^\mu - M_N) \Psi_N - g \Phi \overline{\Psi}_C \Psi_N - g \Phi^* \overline{\Psi}_N \Psi_C - \frac{1}{4} \lambda (\Phi^* \Phi)^2. \tag{1}$$

Note: $\Psi_N$ is electrically neutral, but it’s a Dirac spinor rather than Majorana spinor, so a particle is different from an antiparticle. Just like a neutron is different from an antineutron.

1. First, a few simple questions:
   (a) Are there any renormalizable couplings one may add to the Lagrangian (1) without breaking any symmetries of the theory? Explain your answer.
   (b) Write down the bare Lagrangian of the quantum field theory — including all the counterterms needed to cancel the divergences — and spell out the Feynman rules of the counterterm perturbation theory.
   (c) Some counterterms are related by symmetries and/or Ward identities. Write down all such relations.

2. Second, a lot of hard work: Calculate the UV-infinite parts of all the independent counterterms at the one-loop order of the perturbation theory. Do not bother calculating the UV-finite parts, even if they are IR-divergent — this would take you way too much time. To save more time, use the relations you wrote down in part 1(c): If two or more counterterms are related by a symmetry or a Ward identity, calculate just one of those counterterms, whichever you think is simpler.

For each independent counterterm, start by drawing all the relevant Feynman diagrams. If a diagram was evaluated in class, in my notes, or in a homework, don’t waste your (and the grader’s) time redoing the work, just quote the result and move on to the next diagram. For the remaining diagrams — and there will be plenty of those — use dimensional regularization and work hard. But if some part of the calculation is similar to what I wrote in the solutions or the notes, don’t reproduce my work but simply quote it and adapt it to your needs.
Many diagrams — especially those contributing to the divergence canceled by the $\delta^\lambda$ counterterm — are related by permutations of the external legs. Such symmetries can save you a lot of work, but please be careful counting similar diagrams. Remember that $\Phi$ is different from $\Phi^\ast$.

Note: The on-shell physical amplitudes are gauge invariant, but the off-shell loop diagrams and the counterterms depend on the gauge you work in. To be consistent, you must use the same gauge in all the calculations. Even the diagrams you use to calculate different counterterms must be evaluated in the same gauge.

Hint: many UV divergences cancel out in the Landau gauge $\xi = 0$ for the photon propagators, so this is probably the best gauge to use for this problem. But if you would rather use the Feynman gauge $\xi = 1$ — or any other gauge you like — that’s OK as long as you use the same gauge for all the diagrams.

* For extra credit, allow for an arbitrary gauge parameter $\xi$ and work out how (the infinite parts of) all the counterterms depend on $\xi$.

3. Third, calculate the anomalous dimensions of all the fields and the beta–functions for all the couplings of the theory to the one-loop order. After all the hard work you did in part (2), this should be simple. Remember, at the one-loop level the infinite part of a counterterm determines its dependence on the renormalization energy scale $E$ according to

$$\delta = (\text{overall coefficient}) \times \left( \frac{1}{\epsilon} + \log \frac{\mu^2}{E^2} + O(1) \text{ constant} \right).$$

4. Finally, consider the electromagnetic form factors $F_1(q^2)$ and $F_2(q^2)$ of the neutral fermion $\Psi_N$.

(a) First of all, explain the physical meaning of these form factors and what do they probe. Also, explain why in the $q^2 \to 0$ limit, the $F_1(0)$ form factor must vanish but for the other form factor we expect $F_2(0) \neq 0$.

(b) Now let’s start calculating the $F_1(q^2)$ and the $F_2(q^2)$. Draw all Feynman diagrams contributing to these form factors at the one-loop level. Do any of the counterterms contribute at this level? Explain your answer.

(c) Show that the individual diagrams suffer from logarithmic UV divergences, but the divergences cancel out from the net 1PI amplitude $\Gamma_{1\text{loop}}^\mu$. If they do not, make sure you have not forgotten a diagram and double-check your signs.
(d) Now comes the hard work of evaluating the diagrams: Introduce the Feynman parameters, shift the loop momenta, simplify the numerators, reorganize them according to the form factors $F_1$ and $F_2$, and finally integrate over the loop momenta. Throughout the calculation, keep the fermionic external legs on-shell — including the $\bar{u}(p')\Gamma(\mu)(p',p)u(p)$ context of the amplitude, — but remember that $M_N \neq M_C$. Indeed, allow for general masses $M_N$, $M_C$, and $M_S$ and any off-shell $q^2$.

Some stages of this calculation are going to be similar to what I did in my notes on the electron’s form factors or in the solutions to homework#17, so quote my results and adapt them to your situation instead of redoing my work.

At the end of this part of the problem, you should get formulae of the form

$$\text{neutral } F_1(q^2) = \frac{g^2}{16\pi^2} \sum_i \int d(FP) \text{function}_{1,i}(q^2; FP; \text{masses}; \epsilon),$$

$$\text{neutral } F_2(q^2) = \frac{g^2}{16\pi^2} \sum_i \int d(FP) \text{function}_{2,i}(q^2; FP; \text{masses}; \epsilon),$$

where $FP$ denote the Feynman parameters and $\epsilon = \frac{1}{2}(4 - D)$.

(e) Next, use the results of part (d) to calculate the magnetic dipole moment of the neutral fermion. To simplify the integral over the Feynman parameters, assume $M_S \approx M_C \gg M_N$.

(f) Now consider the electric form factor $F_1(q^2)$ and verify that it duly vanishes for $q^2 = 0$. For this part of the problem, keep the masses general and do not take the $\epsilon \to 0$ limit.

Hint: For $q^2 = 0$, the numerators and the denominators of the diagrams depend on only one Feynman parameter $z$. Moreover, the $\Delta$’s in the denominators of two diagrams are related to each other as $\Delta_1(z_1) = \Delta_2(z_2)$ for $z_2 = 1 - z_1$. Check this relation, then use it to show that

$$F_1(0) = \frac{g^2}{16\pi^2} \times (4\pi\mu^2)\epsilon \Gamma(\epsilon) \times \int_0^1 \frac{d}{dz} \left( \frac{z(1-z)}{[\Delta(z)]^\epsilon} \right) dz = 0.$$

(g) Next, calculate the $F_1(q^2)$ in the limit of very large $(-q^2) \gg$ all masses$^2$ (which allows you to neglect the masses throughout eq. (3)).

(h) Finally, argue that the non-zero result of part (g) implies that the $F_1$ as a function of $q^2$ generally does not vanish, although it does have some discrete zero(s) such as $q^2 = 0$. 
