1. Verify the integrals used by the Feynman’s parameter trick and its generalizations:

\[
\frac{1}{AB} = \int_0^1 \frac{d\xi}{[\xi A + (1 - \xi)B]^2}, \quad (F.a)
\]

\[
\frac{1}{A^n B} = \int_0^1 \frac{n\xi^{n-1}d\xi}{[\xi A + (1 - \xi)B]^{n+1}}, \quad (F.b)
\]

\[
\frac{1}{A^n B^m} = \frac{(n + m - 1)!}{(n-1)!(m-1)!} \times \int_0^1 \frac{\xi^{n-1}(1 - \xi)^{m-1}d\xi}{[\xi A + (1 - \xi)B]^{n+m}}, \quad (F.c)
\]

\[
\frac{1}{ABC} = \int_0^1 \frac{d\xi}{\xi A + \eta B + (1 - \xi - \eta)C}^{\frac{3}{2}} \int_0^1 \frac{2d\eta}{[\xi A + \eta B + (1 - \xi - \eta)C]^3}, \quad (F.d)
\]

\[
\frac{1}{A_1 A_2 \cdots A_k} = \int_{\xi_1,\ldots,\xi_k \geq 0} d^k \xi \delta(\xi_1 + \cdots + \xi_k - 1) \times \frac{(k-1)!}{[\xi_1 A_1 + \cdots + \xi_k A_k]^k}, \quad (F.e)
\]

\[
\frac{1}{A_1^{n_1} A_2^{n_2} \cdots A_k^{n_k}} = \frac{(n_1 + \cdots + n_k - 1)!}{(n_1-1)! \cdots (n_k-1)!} \times \int_{\xi_1,\ldots,\xi_k \geq 0} d^k \xi \delta(\xi_1 + \cdots + \xi_k - 1) \times \frac{\xi_1^{n_1-1} \cdots \xi_k^{n_k-1}}{[\xi_1 A_1 + \cdots + \xi_k A_k]^{n_1+\cdots+n_k}}, \quad (F.f)
\]

2. In class, we have evaluated the one-loop diagram

\[
(1)
\]

using the hard-edge cutoff as an ultraviolet regulator. Your task is to evaluate the same diagram using two other UV regulators: (1) Pauli–Villars, and (2) higher derivatives.
Show that all 3 regulators yield similar amplitudes of the form

\[ \mathcal{M}(\text{diagram (1)}) = \frac{\lambda_{\text{bare}}^2}{32\pi^2} \times \left( \log \frac{\Lambda^2}{m^2} + C - J(t/m^2) + \text{negligible} \right) \]  

(2)

where

\[ J(t/m^2) = \int_0^1 d\xi \log \frac{m^2 - t \times \xi(1 - \xi)}{m^2}, \]  

(3)

‘negligible’ stands for terms that vanish as negative powers of the cutoff scale \( \Lambda \) for \( \Lambda \to \infty \), and \( C \) is an \( O(1) \) numeric constant that depends on the particular UV regulator:

\[ C_{\text{hard edge}} \neq C_{\text{Paili Villars}} \neq C_{\text{higher derivative}}. \]  

(4)

Fortunately, this regulator dependence can be canceled by adjusting the cutoff scale parameter \( \Lambda \) for each regulator: Let

\[ \Lambda_{\text{HE}}^2 \times e^{C_{\text{HE}}} = \Lambda_{\text{PV}}^2 \times e^{C_{\text{PV}}} = \Lambda_{\text{HD}}^2 \times e^{C_{\text{HD}}}, \]  

(5)

then all 3 regulators would yield exactly the same loop amplitude (2).

Note: the dimensional regularization also yields exactly the same amplitude (2), provided we identify the UV cutoff scale as

\[ \Lambda_{\text{DR}}^2 = \mu^2 \times \exp \left( \frac{1}{\epsilon} = \frac{2}{4 - D} \right) \]  

(6)

and then set

\[ \Lambda_{\text{DR}}^2 \times e^{C_{\text{DR}}} = \Lambda_{\text{HE}}^2 \times e^{C_{\text{HE}}} = \Lambda_{\text{PV}}^2 \times e^{C_{\text{PV}}} = \Lambda_{\text{HE}}^2 \times e^{C_{\text{HD}}} \]  

(7)

for a suitable \( O(1) \) numeric constant \( C_{\text{DR}} \).
Hint: for the higher-derivative regulator, approximate the modified propagator as

$$\frac{i}{q^2 - m^2 - (q^4/\Lambda^2) + i\epsilon} \approx \frac{i}{q^2 - m^2 + i\epsilon} \times \frac{-\Lambda^2}{q^2 - \Lambda^2 + i\epsilon}$$

(8)

where the second factor differs from 1 only for very large momenta. Consequently, for the two propagators in the loop we may further approximate

$$\frac{-\Lambda^2}{q_1^2 - \Lambda^2 + i\epsilon} \approx \frac{-\Lambda^2}{q_2^2 - \Lambda^2 + i\epsilon} \approx \frac{-\Lambda^2}{(q_1 - \xi q_{net})^2 - \Lambda^2 + i\epsilon}$$

(9)