1. Consider the axial anomaly in a non-abelian gauge theory, for example QCD with $N_f$ massless quark flavors,

$$ J_A^\mu = \sum_{i,f} \bar{\Psi}_{if} \gamma^5 \gamma^\mu \Psi_{if}, \quad \partial_\mu J_A^\mu = -\frac{N_f g^2}{16\pi^2} \epsilon^{\alpha\beta\mu\nu} \text{tr}\left( F_{\alpha\beta} F_{\mu\nu} \right) $$  

where $F_{\mu\nu}$ is the non-abelian gauge field strength.

(a) Expand the right hand side of eq. (1) into 2–gluon, 3–gluon, and 4–gluon terms and show that the 4–gluon term vanishes identically.

Hint: Use the cyclic symmetry of the trace.

The two-gluon anomaly term obtains from the triangle diagrams

\[ \text{triangle diagrams with gluon permutation.} \]

This works exactly as discussed in class for the QED, except in QCD we should trace $F_{\alpha\beta} F_{\gamma\delta}$ over the quark colors and flavors. But in QCD there is also the three-gluon anomaly (cf. part (a)) which obtains from the quadrangle diagrams

\[ \text{quadrangle diagrams with gluon permutations.} \]

Since the quadrangle diagrams suffer from linear UV divergences, we need to regulate them, so let's use the Pauli–Villars regulator.
(b) Show that

\[ i q_\alpha \times \gamma^\alpha \gamma^5 \text{ regulated } = -2i M \gamma^5 \text{ Pauli–Villars compendator only } \]

+ terms which cancel after summing over gluon permutations

(c) Finally, evaluate the the quadrangle diagrams for the Pauli–Villars regulators and derive the three-gluon anomaly in QCD.

2. Next, a reading assignment: §22.2–3 of \textit{Weinberg} about the chiral anomaly. Pay particular attention to the Jacobian of the fermion path integral and to regularization of the functional trace.

3. In any \textit{even} spacetime dimension \( d = 2n \), a massless Dirac fermion has an axial symmetry \( \Psi(x) \rightarrow \exp(i\theta \Gamma)\Psi(x) \) where \( \Gamma \) generalizes the \( \gamma^5 \). Classically, the axial current \( J^\mu_A = \overline{\Psi} \Gamma \gamma^\mu \Psi \) is conserved, but when the fermion is coupled to a gauge field — abelian or non-abelian — the axial symmetry is broken by the anomaly and

\[
\partial_\mu J^\mu_A = -\frac{2}{n!} \left(\frac{g}{4\pi}\right)^n \epsilon^{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n} \text{tr}(F_{\alpha_1\beta_1}F_{\alpha_2\beta_2}\cdots F_{\alpha_n\beta_n}).
\]

(5)

Generalize Weinberg’s calculation of the anomaly via Jacobian of the fermionic path integral to any even spacetime dimension \( d = 2n \).

For your information, in \( 2n \) Euclidean dimensions \( \{\gamma^\mu, \gamma^\nu\} = +2\delta^{\mu\nu} \), \( \Gamma = i^{n-2}\gamma^1\gamma^2\cdots\gamma^{2n} \), \( \{\Gamma, \gamma^\mu\} = 0 \), \( \Gamma^2 = +1 \), and for any \( 2n = d \) matrices \( \gamma^\alpha, \ldots, \gamma^\omega \), \( \text{tr}(\Gamma\gamma^\alpha\gamma^\beta\cdots\gamma^\omega) = 2^n t^{2n} e^{\alpha\beta\cdots\omega} \).