Problem 1(a):
Let’s start with a simpler problem: two concentric metal spheres, and the space between the spheres is either completely empty or completely filled with a uniform dielectric. By spherical symmetry of this setup, the electric field between the spheres must point in the radial direction while its magnitude depends only on the radius; by Gauss Law,

\[ \mathbf{E}(x) = \frac{A}{r^2} \mathbf{n} \]  

(S.1)

for some constant \( A \). The value of this constant follows from the voltage between the plates:

\[ V = \Phi(a) - \Phi(b) = A \left( \frac{1}{a} - \frac{1}{b} \right) \implies A = \frac{V_{ab}}{b-a}. \]  

(S.2)

For the problem at hand, the space between the spheres is half-filled with a dielectric while the other half is vacuum. Fortunately, each material occupies a hemisphere, so the boundary between them lies in the equatorial plane. Consequently, for this geometry the electric field is exactly as in eq. (S.1), while the displacement field \( \mathbf{D} \) is

\[ \mathbf{D}_{\text{in vacuum}} = \varepsilon_0 \mathbf{E} = \frac{\varepsilon_0 A}{r^2} \mathbf{n}, \]

\[ \mathbf{D}_{\text{in dielectric}} = \varepsilon \varepsilon_0 \mathbf{E} = \frac{\varepsilon \varepsilon_0 A}{r^2} \mathbf{n}. \]  

(S.3)

Graphically,

Indeed, the field (S.1) and (S.3) obey the boundary conditions at the dielectric-boundary
interface

\[ \mathbf{E}^\parallel_{\text{vac}} = \mathbf{E}^\parallel_{\text{diel}}, \quad \mathbf{D}^\perp_{\text{vac}} = \mathbf{D}_{\text{diel}} \]  \hspace{1cm} (S.5)

since (1) the \( \mathbf{E} \) field is completely continuous across the boundary, while (2) the \( \mathbf{D} \) field at both sides of the boundary points in the radial direction which happens to be parallel to the boundary, thus

\[ \mathbf{D}^\perp_{\text{vac}} = 0 = \mathbf{D}_{\text{diel}}. \]  \hspace{1cm} (S.6)

The remaining conditions on the \( \mathbf{E} \) and \( \mathbf{D} \) fields: the equations of state in the vacuum and in the dielectric,

\[ \nabla \times \mathbf{E} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{D} = \rho = 0 \]  \hspace{1cm} (S.7)

in both halves of the space between the spheres, and the boundary conditions on the metal spheres themselves —

for \( x \in \text{inner sphere} \), \( \Phi(x) = \text{const} \),

for \( x \in \text{outer sphere} \), \( \Phi(x) = \text{const} \),

\[ \Phi(\text{inner sphere}) - \Phi(\text{outer sphere}) = V, \]  \hspace{1cm} (S.8)

— are also obviously satisfied.

Thus, the electric tension and displacement fields for this problem are indeed as in eqs. (S.1) and (S.3).

Problem 1(b):
Inside the metal of each sphere \( \mathbf{E} = 0 \) and hence \( \mathbf{D} = 0 \). This makes the \( \mathbf{D} \) field discontinuous at the outer surface of the inner sphere and at the inner surface of the outer sphere, and the physical reason for such a discontinuity is the surface density \( \sigma \) of macroscopic charges. By the Gauss Law,

\[ \sigma = \mathbf{D}^\perp(\text{just outside the metal}) = \mathbf{D} \cdot \mathbf{n}^\perp \]  \hspace{1cm} (S.9)

where \( \mathbf{n}^\perp \) is the unit vector normal to the metal's surface and point out from the metal. For the outer surface of the inner sphere this makes \( \mathbf{n}^\perp = +\mathbf{n}_r \) but for the inner surface of the
outer sphere $\mathbf{n}^\perp = -\mathbf{n}_r$. Consequently,

\begin{align*}
\sigma(\text{inner sphere, vacuum side}) &= +\frac{\epsilon_0 A}{a^2}, \\
\sigma(\text{inner sphere, dielectric side}) &= +\frac{\epsilon \epsilon_0 A}{a^2}, \\
\sigma(\text{outer sphere, vacuum side}) &= -\frac{\epsilon_0 A}{b^2}, \\
\sigma(\text{outer sphere, dielectric side}) &= -\frac{\epsilon \epsilon_0 A}{b^2},
\end{align*}
(S.10, S.11, S.12, S.13)

where $A$ is as in eq. (S.2).

Given these surface charge densities, the net charge on the inner sphere is

\begin{align*}
Q_{\text{inner}} &= +\frac{\epsilon_0 A}{a^2} \times 2\pi a^2 + \frac{\epsilon \epsilon_0 A}{a^2} \times 2\pi a^2 = 2\pi(\epsilon + 1)\epsilon_0 A,
\end{align*}
(S.14)

while the net charge on the outer sphere is

\begin{align*}
Q_{\text{outer}} &= -\frac{\epsilon_0 A}{b^2} \times 2\pi b^2 - \frac{\epsilon \epsilon_0 A}{b^2} \times 2\pi b^2 = -2\pi(\epsilon + 1)\epsilon_0 A = -Q_{\text{inner}}. 
\end{align*}
(S.15)

Treating these two metal spheres as plates of a capacitor with charges $\pm Q$, the capacitance of this capacitor is

\begin{align*}
C &= \frac{Q}{V} = 2\pi(\epsilon + 1)\epsilon_0 \times \frac{A}{V} = 2\pi(\epsilon + 1)\epsilon_0 \times \frac{ab}{b-a}.
\end{align*}
(S.16)

Problem 3(a):
Let’s span the current-carrying wire loop $\mathcal{L}$ with same surface $\mathcal{S}$. To find the solid angle occupied by the image of $\mathcal{S}$ as viewed from point $x$, we project $\mathcal{S}$ onto a unit sphere centered on $x$, and then measure the area of the image. For an infinitesimal piece of $\mathcal{S}$ of vector area $da$, we first project this piece onto a line of sight from $x$ and then further project it onto the unit sphere:
On this picture

\[ \Delta A_{2n} = \Delta A_2 \times \cos \theta, \quad \Delta \Omega = \frac{\Delta A_1}{r_1^2} = \frac{\Delta A_2}{r_2^2}, \quad \text{(S.17)} \]

which in our notations corresponds to

\[ d\Omega = \frac{n \cdot d^2\text{area}}{R^2} \quad \text{(S.18)} \]

where \( R \) is the distance from the observation point \( x \) and \( n \) is the unit vector along the line of sight. For the infinitesimal piece of \( S \) located at \( y \),

\[ R = |y - x|, \quad n = \frac{y - x}{|y - x|}, \quad \text{(S.19)} \]

hence

\[ d\Omega = \frac{(y - x) \cdot d^2\text{area}(y)}{|y - x|^3}. \quad \text{(S.20)} \]

Integrating this formula over the whole surface \( S \) spanning the loop \( \mathcal{L} \), we arrive at

\[ \Omega(x) = \int \int_S \frac{(y - x) \cdot d^2\text{area}(y)}{|y - x|^3}. \quad \text{(2)} \]

*Quod erat demonstrandum.*
Problem 3(b):
The sign convention for the $\Omega(x)$ follow from eq. (2) and the standard convention for the direction of the area vector. For simplicity, consider a flat loop $\mathcal{L}$ spanned by a flat surface $\mathcal{S}$. The area vector $\mathbf{a}$ of this surface is perpendicular to the surface itself, but which perpendicular? To make the Stokes’ theorem work without an extra sign, the direction of $\mathbf{a}$ should follow from the sense of the loop $\mathcal{L}$ by the right hand rule: if you see the loop (or rather the current in the loop) running clockwise, then the area vector $\mathbf{a}$ points away from you, $i.e.$, makes angle $< 90^\circ$ with the line of sight; but if you see the loop $\mathcal{L}$ running counterclockwise, then $\mathbf{a}$ points towards you, $i.e.$, makes angle $> 90^\circ$ with the line of sight. The same rule applies to the area vector of any infinitesimal piece of $\mathcal{S}$, so the integrand in eq. (2) is positive for a clockwise loop $\mathcal{L}$ and negative for a counterclockwise $\mathcal{L}$.

For a non-flat surface, the rule for for the direction of the $d\mathbf{a}$ vector is topological. The surface $\mathcal{S}$ spanning the loop $\mathcal{L}$ must be orientable, $i.e.$, have two well defined sides; Möbius strips and similar non-orientable surfaces are not allowed. Depending on the sense of the loop $\mathcal{L}$, we call one side ‘inner’ and the other side ‘outer’ according at the right hand rule, and then the direction of $d\mathbf{a}$ is the $\perp$ to the surface (at the point in question) and pointing from the ‘inside’ to the ‘outside’. Consequently, if the loop $\mathcal{L}$ and the surface $\mathcal{S}$ are not too twisted and lie largely to one side of $\mathbf{x}$, then the sign of $\Omega(x)$ obtaining from eq. (2) follows from the sense of the loop as viewed from $\mathbf{x}$ similarly to the flat-surface case.

The problem with eq. (2) is that different surfaces spanning the same loop $\mathcal{L}$ may yield different values of $\Omega(x)$, although all the different values for the same point $\mathbf{x}$ differ by $4\pi$, or at worse by $4\pi \times$ an integer. To see how this works, let two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ span $\mathcal{L}$ and consider the space $\mathcal{V}$ trapped between these surfaces. Together, $\mathcal{S}_1$ and $\mathcal{S}_2$ form the complete surface of the volume $\mathcal{V}$, but one of the the two surfaces — say, $\mathcal{S}_2$ — has a wrong orientation — its infinitesimal area vectors point inside $\mathcal{V}$ rather than outside. So properly speaking, the complete surface of $\mathcal{V}$ is $\mathcal{S}_1 - \mathcal{S}_2$. By Gauss theorem, this means that for any vector field $\mathbf{f}(y)$

$$\iiint_{\mathcal{V}} \nabla \cdot \mathbf{f} d^3y = \iint_{\mathcal{S}_1} \mathbf{f} \cdot d^2\mathbf{a} - \iint_{\mathcal{S}_2} \mathbf{f} \cdot d^2\mathbf{a}. \quad (S.21)$$
Now let
\[ f(y) = \frac{(y - x)}{|y - x|^3} \] (S.22)
for any fixed point \( x \). Then calculating \( \Omega(x) \) using the surfaces \( S_1 \) and \( S_2 \) and taking the difference, we obtain
\[
\Omega_1(x) - \Omega_2(x) = \iiint_{S_1} \frac{(y - x) \cdot d^2a(y)}{|y - x|^3} - \iiint_{S_2} \frac{(y - x) \cdot d^2a(y)}{|y - x|^3} = \iiint_V \nabla_y \left( \frac{(y - x)}{|y - x|^3} \right) \, d^3y.
\] (S.23)

But
\[
\nabla_y \cdot \left( \frac{(y - x)}{|y - x|^3} \right) = 4\pi \delta^{(3)}(x - y),
\] (S.24)
hence
\[
\Omega_1(x) - \Omega_2(x) = \begin{cases} 
4\pi & \text{if } x \text{ lies inside } V, i.e. \text{ between } S_1 \text{ and } S_2, \\
0 & \text{otherwise.}
\end{cases}
\] (S.25)

In other words, if we take two surfaces spanning the same loop \( L \) but on different sides from point \( x \), then the corresponding angles \( \Omega_1(x) \) and \( \Omega_2(x) \) differ by \( 4\pi \).

A qualitative way to see this multivaluedness is to project both surfaces \( S_1 \) and \( S_2 \) and the loop \( L \) onto the the unit sphere centered at \( x \). The image of the loop \( L \) divides the sphere into two parts, and if the surfaces \( S_1 \) and \( S_2 \) lie on different sides of \( x \), then their images are precisely the two parts of the sphere divided by the image of \( L \). Together, these two images complete the sphere, so their solid angles must add up to \( 4\pi \). But one of the two images has a wrong orientation, so the solid angle it occupies should be taken with a minus sign, hence
\[
\text{either } \Omega_1(x) - \Omega_2(x) = 4\pi \text{ or } \Omega_2(x) - \Omega_1(x) = 4\pi.
\] (S.26)

Finally, when the wire loop \( L \) is a coil of many turns, a surface spanning it must span every turn, which calls for some kind of a helicoid. Projecting such a helicoid onto a sphere creates many overlapping patches, and their solid angles must be added up to produce the correct \( \Omega(x) \). Consequently, for an \( x \) close to a coil of many turns we may get \( \Omega(x) \gg 4\pi \).
Also, when $x$ is in the middle of the coil, then different helicoid-like surfaces spanning the same coil may have several turns on different side of $x$. Consequently, the values of $\Omega(x)$ for these two surfaces may differ not just by $4\pi$ but by $4\pi \times $ integer, \textit{i.e.},

$$\Omega_1(x) - \Omega_2(x) = 0 \text{ or } \pm 4\pi \text{ or } \pm 8\pi \text{ or } \pm 12\pi \text{ or } \cdots.$$ (S.27)

However, since the differences between the values of $\Omega(x)$ for the same point $x$ are always integer multiples of $4\pi$, they cannot gradually change from $x$ to $x + \delta x$. Therefore, \textit{despite the multivaluedness of the $\Omega(x)$, the gradient $\nabla\Omega(x)$ is single-valued.}

**Problem 3(c):**
First, let’s derive eq. (3). Take any vector field $f(y)$ and any constant vector $c$. By the double vector product formula,

$$\nabla \times (f \times c) = (c \cdot \nabla)f - (\nabla \cdot f)c.$$ (S.28)

In particular, let

$$f(y) = \frac{(y-x)}{|y-x|^3} = \nabla_y \left( -\frac{1}{|y-x|} \right)$$ (S.29)

for a fixed $x$. For this ‘field’, $\nabla_y \cdot f = 0$ for $y \neq x$, so eq. (S.28) simplifies to

$$\nabla_y \times (f \times c) = (c \cdot \nabla_y)f$$ (S.30)

and hence

$$\nabla_y \times \left( \frac{(y-x)}{|y-x|^3} \times c \right) = (c \cdot \nabla_y) \frac{(y-x)}{|y-x|^3}.$$ (3)

Now let’s use this formula to calculate the gradient of $\Omega(x)$ as calculated in eq. (2). Let
c be come constant vector, then

\[ \mathbf{c} \cdot \nabla \Omega(\mathbf{x}) = (\mathbf{c} \cdot \nabla_x) \iint_S \frac{(\mathbf{y} - \mathbf{x})}{|\mathbf{y} - \mathbf{x}|^3} \cdot d^2 \mathbf{a}(\mathbf{y}) = \iint_S (\mathbf{c} \cdot \nabla_x) \left( \frac{(\mathbf{y} - \mathbf{x})}{|\mathbf{y} - \mathbf{x}|^3} \right) \cdot d^2 \mathbf{a}(\mathbf{y}) \]

\[ \langle \text{using } \nabla_x f(\mathbf{y} - \mathbf{x}) = -\nabla_y f(\mathbf{y} - \mathbf{x}) \rangle \]

\[ = - \iint_S (\mathbf{c} \cdot \nabla_y) \left( \frac{(\mathbf{y} - \mathbf{x})}{|\mathbf{y} - \mathbf{x}|^3} \right) \cdot d^2 \mathbf{a}(\mathbf{y}) \]

\[ = - \iint_S \left( \nabla \times \left( \frac{(\mathbf{y} - \mathbf{x})}{|\mathbf{y} - \mathbf{x}|^3} \times \mathbf{c} \right) \right) \cdot d^2 \mathbf{a}(\mathbf{y}) \quad \langle \text{using eq. (3)} \rangle \]

\[ = - \oint_L \frac{(\mathbf{y} - \mathbf{x})}{|\mathbf{y} - \mathbf{x}|^3} \times \mathbf{c} \cdot d^2 \mathbf{y} \quad \langle \text{by the Stokes' theorem} \rangle \]

\[ = + \oint_L (\mathbf{c} \cdot (\frac{(\mathbf{y} - \mathbf{x})}{|\mathbf{y} - \mathbf{x}|^3} \times d\mathbf{y})) \quad \langle \text{vector identity} \rangle \]

\[ = \mathbf{c} \cdot \oint_L \frac{(\mathbf{y} - \mathbf{x})}{|\mathbf{y} - \mathbf{x}|^3} \times d\mathbf{y}. \]  

(S.31)

Since \( \mathbf{c} \) on both sides of this equation is an arbitrary constant vector, this means

\[ \nabla \Omega(\mathbf{x}) = \oint_L \frac{(\mathbf{y} - \mathbf{x})}{|\mathbf{y} - \mathbf{x}|^3} \times d\mathbf{y} = - \oint_L d\mathbf{y} \times \frac{(\mathbf{y} - \mathbf{x})}{|\mathbf{y} - \mathbf{x}|^3}. \]  

(S.32)

Finally, let’s see what all this math has to do with eq. (1) for the scalar magnetic potential \( \Psi(\mathbf{x}) \). The magnetic intensity field \( \mathbf{H} \) follows from \( \Psi(\mathbf{x}) \) as \(-\nabla \Psi\), hence according to eqs. (1) and (S.32),

\[ \mathbf{H}(\mathbf{x}) = -\nabla \Psi(\mathbf{x}) = -\frac{I}{4\pi} \nabla \Omega(\mathbf{x}) = + \frac{1}{4\pi} \oint_L \mathbf{I} \times \frac{(\mathbf{y} - \mathbf{x})}{|\mathbf{y} - \mathbf{x}|^3}. \]  

(S.33)

But this is precisely the Biot–Savart–Laplace formula for the magnetic field of the current \( I \) flowing through the wire loop \( \mathcal{L} \)!

*Quod erat demonstrandum.*
Problem 4(a):
Before addressing the problem at hand, let’s consider work and energy of a variable-capacitance capacitor connected to a battery. As the capacitance changes, the charge stored in the capacitor changes, so a current flows through the battery, which performs electric work

\[ W_{el} = V \delta Q. \]

Also, changing the capacitance of a charged capacitor takes a mechanical work \( W_{\text{mech}} \), which can be calculated from the energy balance equation

\[ \delta U = \delta W_{el} + \delta W_{\text{mech}} \]  \hspace{1cm} (S.34)

where

\[ U = \frac{Q^2}{2C} = \frac{CV^2}{2} = \frac{VQ}{2} \]  \hspace{1cm} (S.35)

is the energy stored in the capacitor. Consequently

\[ \delta U = \frac{Q\delta Q}{C} - \frac{Q^2}{2C^2} \delta C = V \delta Q - \frac{V^2}{2} \delta C, \]  \hspace{1cm} (S.36)

and hence

\[ \delta W_{\text{mech}} = \delta U - V \delta Q = -\frac{V^2}{2} \delta C. \]  \hspace{1cm} (S.37)

BTW, the above calculation does not depend on the battery’s voltage \( V \) being fixed. So the mechanical work involved in an infinitesimal change of capacitance is always given by eq. (S.37), regardless of whether the capacitor is hooked up to a fixed-voltage battery, or to more complicated power supply, or even charged and disconnected.

Now consider moving a piece of dielectric in or out from between the plates of a charged capacitor. Such movement changes the capacitance \( C \), and according to eq. (S.37) this takes a mechanical work and hence mechanical forces. Specifically, there is a force pulling the dielectric inside the capacitor.
To see how this works, take a parallel plate capacitor, with rectangular plates of length $L$, width $w$, and distance $d$ between the plates, $d \ll L, w$. The movable dielectric completely fills the gap between the plates and covers their whole width but not the length:

This capacitor can be thought as a parallel circuit of two capacitors, one vacuum-filled of length $L - z$ and the other dielectric-filled of length $x$, so altogether

$$C = \varepsilon \varepsilon_0 \frac{w x}{d} + \varepsilon_0 \frac{(L - x) w}{d}.$$ (S.38)

Pulling the dielectric in through length $\delta x$ changes the capacitance by

$$\delta C = (\varepsilon - 1) \varepsilon_0 \frac{w}{d} \times \delta x,$$ (S.39)

and according to eq. (S.37) this takes mechanical work upon the capacitor

$$W_{\text{mech}} = - \frac{V^2}{2} \times \frac{(\varepsilon - 1) \varepsilon_0 w}{d} \times \delta x.$$ (S.40)

The mechanical work done by the capacitor obtains by sign reversal, and equating this work to $F \times \delta x$, we find the force $F$ pulling the dielectric inside the capacitor,

$$F = + \frac{V^2}{2} \times \frac{(\varepsilon - 1) \varepsilon_0 w}{d}.$$ (S.41)

Finally, let’s turn the capacitor plates vertically and immerse them part-way into transformer oil. The oil is a dielectric, so the force (S.41) pulls it into the space between the plates, and that’s what raises the oil level between the plates compared to its level outside. The
height \( h \) through which the oil is raised follows from balancing the pulling force \( F \) against the weight of extra oil between the plates,

\[
F = g\rho wdh,
\]

and hence

\[
h = \frac{F}{g\rho w} = (\epsilon - 1)\epsilon_0 \frac{V^2}{2g\rho d^2}.
\]

Note that the plates’ width \( w \) cancels out from this formula.

For a numeric example, take transformer oil with dielectric constant \( \epsilon = 1.34 \) and mass density 882 kg/m\(^3\), make the gap between the plates 1.00 mm wide, and charge the capacitor to \( V = 3000 \) Volts, then the oil in the gap will rise to \( h = 4.6 \) mm.