Problem 1(a):
In terms of the 3D retarded Green’s function

\[ G_R(x - y, t_x - t_y) = \frac{\delta(t_x - t_y - r/c)}{4\pi r} \quad \text{where } r = |x - y|, \quad (S.1) \]

the wave generated by an instant line source at time \( t_y = 0 \) is simply

\[ \Psi(x, t) = \int_{\text{line}} dl_y G(x - y, t). \quad (S.2) \]

For our purposes, let the source line be the whole \( x_3 \) axis, then

\[ \Psi(x, t) = \int_{-\infty}^{+\infty} dy_3 \frac{\delta(t - r/c)}{4\pi r} = \sum_{\text{points } y_3 \text{ where } r = ct} \frac{c}{4\pi r} \left/ \left| \frac{\partial r_{y_3}}{\partial y_3} \right| \right. \quad (S.3) \]

The solutions to the \( r = ct \) condition depend on \( r_{2d} = \sqrt{x_1^2 + x_2^2} \): For \( r_{2d} > ct \) there are no solutions, while for \( r_{2d} < ct \) there are two solutions at \( y_3 = x_3 \pm \sqrt{c^2t^2 - r_{2d}^2} \), at which points

\[ \frac{\partial r}{\partial y_3} = \pm \frac{c^2t^2 - r_{2d}^2}{r = ct} \Rightarrow \frac{c}{4\pi r} \left/ \left| \frac{\partial r_{y_3}}{\partial y_3} \right| \right. = \frac{c}{4\pi \sqrt{c^2t^2 - r_{2d}^2}}. \quad (S.4) \]

Consequently,

\[ \Psi(x, t) = \frac{2c\Theta(ct - r_{2d})}{4\pi \sqrt{c^2t^2 - r_{2d}^2}}, \quad (S.5) \]

in perfect agreement with eq. (2) for the 2D wave of a point source.
Problem 1(b):
This time, the instant source spans an entire plain, which we take to be the \((y_2, y_3)\) plane, so the wave generated by this source is

\[
\Psi(x, t) = \int dy_2 dy_3 \frac{\delta(t - r/c)}{4\pi r} = \int_0^\infty ds 2\pi s \frac{c\delta(ct - r(s))}{4\pi r(s)}
\]

for

\[
s = \sqrt{(y_2 - x_2)^2 + (y_3 - x_2)^2} \quad \text{and} \quad r = \sqrt{x_1^2 + s^2}.
\]

For \(|x_1| > ct\) there are no \(s\) such that \(r(s) = ct\) and hence \(\Psi = 0\), while for \(|x_1| < ct\) there is one such point \(s = +\sqrt{c^2t^2 - x_1^2}\) and hence

\[
\Psi = \frac{2\pi sc}{4\pi r} / \frac{\partial r}{\partial s} = \frac{sc}{2r} / \frac{s}{r} = \frac{c}{2}.
\]

Altogether,

\[
\Psi = \frac{c}{2} \Theta(ct - |x_1|),
\]

in perfect agreement with eq. (3) for the 1D wave of a point source.

Problem 2(a):
The densities (6) of charges and currents trivially obey the continuity equation:

\[
\nabla \cdot J = \delta'(t) \nabla \cdot (p \delta^{(3)}(x)) = \delta'(t) (p \cdot \delta)\delta^{(3)}(x) = -\frac{\partial \rho}{\partial t}.
\]

As to the scalar potential in the Coulomb gauge, the formal solution of the first eq. (4) is

\[
\Phi = \frac{1}{\epsilon_0} - \frac{1}{\nabla^2} \rho = -\frac{1}{\epsilon_0} \delta(t) \frac{-1}{\nabla^2} (p \cdot \nabla)\delta^{(3)}(x).
\]

Since the operators \((p \cdot \nabla)\) and \((-1/\nabla^2)\) commute with each other, we have

\[
-\frac{1}{\nabla^2} (p \cdot \nabla)\delta^{(3)}(x) = (p \cdot \nabla) \frac{-1}{\nabla^2} \delta^{(3)}(x) = (p \cdot \nabla) \frac{1}{4\pi r} = -\frac{p \cdot n}{4\pi r^2}
\]

and consequently

\[
\Phi(x, t) = +\delta(t) \frac{p \cdot n}{4\pi \epsilon_0 r^2}.
\]
Problem 2(b):
For the current density as in eq. (6),
\[
\nabla \left( -\frac{1}{\nabla^2} (\nabla \cdot J) \right) = \delta'(t) \nabla -\frac{1}{\nabla^2} (p \cdot \nabla) \delta^{(3)}(x),
\]
where all three operators — \(\nabla\), \((p \cdot \nabla)\), and \((-1/\nabla^2)\) — commute with each other. Consequently,
\[
\nabla -\frac{1}{\nabla^2} (p \cdot \nabla) \delta^{(3)}(x) = \nabla (p \cdot \nabla) -\frac{1}{\nabla^2} \delta^{(3)}(x) = \nabla (p \cdot \nabla) \frac{1}{4\pi r},
\]
and hence eq. (7) for the transverse current. As to eq. (8), it follows from eq. (7) and
\[
\nabla_i \nabla_j \frac{1}{4\pi r} = \frac{3n_i n_j - \delta_{ij}}{4\pi r^3} - \frac{\delta_{ij}}{3} \delta^{(3)}(x).
\]

Problem 2(c):
Let’s start with the first lemma (9). In spherical coordinates \((r, \theta, \phi)\) for \(z\), the LHS of eq. (9) — which I am going to denote \(L_1\) — becomes
\[
L_1 \triangleq \int \int \left( \int \frac{d^3z}{|z|} \frac{\delta'(t - |z|/c)}{r} \times F(z) \right) = \int \int \left( \int_0^\infty r^2 \Omega(\theta, \phi) \frac{\delta'(t - r/c)}{r} \times F(r, \theta, \phi) \right) \times F(r, \theta, \phi)
\]
\[
= \int \int \left( \int_0^\infty r \delta'(t - r/c) \times F(r, \theta, \phi) \right) d\Omega(\theta, \phi) \times F(r, \theta, \phi)
\]
where
\[
G(t) \triangleq \int \Omega(\theta, \phi) F(r, \theta, \phi).
\]
Next,
\[
\delta'(t - r/c) = -c^2 \delta'(r - ct),
\]
and consequently
\[
L_1 = -c^2 \int_0^\infty dr \delta'(r - ct) \times r G(r) = +c^2 \int_0^\infty dr \delta(r - ct) \times \frac{d}{dr}(r G(r)).
\]
procedure for handling the derivatives of the delta functions. The remaining integral involving the ordinary delta function $\delta(r - ct)$ yields

$$L_1 = +c^2 \left[ \frac{d}{dr}(rG(r)) = \left( 1 + r \frac{d}{dr} \right) G(r) \right]_{r = ct}$$

(S.20)

provided $r = ct$ is within the integration range $0 < ct < \infty$, but zero otherwise. In other words,

$$L_1 = c^2 \Theta(t) \left[ \left( 1 + r \frac{d}{dr} \right) G(r) \right]_{r = ct}$$

(S.21)

where $\Theta(t)$ is the step-function: 1 for $t > 0$ but 0 for $t < 0$. Finally, plugging eq. (S.17) for $G(r)$ into eq. (S.21) completes the proof of the first Lemma (9).

As to the second Lemma (10), it’s the good old Mean Value Theorem of electrostatics: Averaging a Coulomb potential of a point charge over a spherical surface yields the potential at the sphere’s center provided the charge is outside the sphere. For the charge inside the sphere, the averaging yields a potential of a similar charge moved to the center of the sphere.

The simplest way to prove the mean value theorem is via the multipole expansion (once you know how it works),

$$\frac{1}{|x + Rn|} = \sum_{\ell = 0}^{\infty} \frac{\min(|x|, R)^\ell}{\max(|x|, R)^{\ell+1}} \times P_\ell(-\cos \theta)$$

(S.22)

where $\theta$ is the angle between the unit vector $n$ and the vector $x$. Let’s plug the expansion (S.22) into the angular integral (10) and integrate term by term: without the radial factor,

$$\frac{1}{4\pi} \int d^2 \Omega_n P_\ell(-\cos \theta) = \frac{1}{2} \int_{-1}^{+1} d \cos \theta P_\ell(-\cos \theta) = \delta_{\ell,0},$$

(S.23)

so only the $\ell = 0$ term contributes to the net integral. Consequently,

$$\frac{1}{4\pi} \int d^2 \Omega_n \frac{1}{|x + Rn|} = \frac{\min(|x|, R)^0}{\max(|x|, R)^1} = \frac{1}{\max(|x|, R)}.$$

(S.24)

which completes the proof of the mean value theorem (10).
Problem 2(d):

The formal solution of the wave equation (4) for the vector potential obtains via the retarded Green’s function as

$$A(x, t_x) = \frac{\mu_0}{4\pi} \iiint_{\text{whole space}} d^3y \int dt_y \frac{\delta(t_x - t_y - |x - y|/c)}{|x - y|} J_T(y, t_y)$$  \hspace{1cm} (S.25)$$

where $J_T$ is the transverse current (7). Since the time-dependence and the $y$ dependence of this transverse current factorize as

$$J_T(y, t_y) = \delta'(t_y) \times J_0(y),$$  \hspace{1cm} (S.26)$$

the time integral in eq. (S.25) yields

$$\int dt_y \delta(t_x - t_y - |x - y|/c) \times \delta'(t_y) = \delta'(t_x - |x - y|/c)$$  \hspace{1cm} (S.27)$$

and hence

$$A(x, t) = \frac{\mu_0}{4\pi} \iiint_{\text{whole space}} d^3y \frac{\delta'(t_x - |y - x|/c)}{|y - x|} J_0(y)$$  \hspace{1cm} (S.28)$$

for

$$J_0(y) = p \delta^{(3)}(y) + \nabla(p \cdot \nabla) \frac{1}{4\pi |y|}.$$  \hspace{1cm} (S.29)$$

Plugging the first term in this current into the integral (S.28) yields

$$A_1(x, t_x) = \frac{\mu_0 p}{4\pi |x|} \delta'(t_x - |x|/c),$$  \hspace{1cm} (S.30)$$

a flash spreading out in all directions at the speed of light. This flash vanishes for $t < |x|/c$, so we may ignore it for this part of the problem, although we will need it later in part (f).
For the moment, let’s focus on the second terms in the transverse current and plug it into the integral (S.28), thus

\[
A_2(x, t_x) = \frac{\mu_0}{4\pi} \iiint_{\text{whole space}} d^3 y \frac{\delta'(t_x - |y - x|/c)}{|y - x|} \nabla_y (p \cdot \nabla_y) \frac{1}{4\pi |y|}.
\] (S.31)

To evaluate this integral, let’s change the integration variable from \(y\) to \(z = y - x\), thus

\[
A_2(x, t_x) = \frac{\mu_0}{4\pi} \iiint_{\text{whole space}} d^3 z \frac{\delta'(t_x - |z|/c)}{|z|} \nabla_y (p \cdot \nabla_y) \frac{1}{4\pi |x + z|}.
\] (S.32)

Formally, the space derivatives inside this integral are WRT \(y = x + z\), but since they act on a function which depends only on the \(y = x + z\), we may change them to \(x\)-derivatives for a fixed \(z\),

\[
\left[ \nabla_y (p \cdot \nabla_y) \frac{1}{4\pi |y|} \right]_{y = x + z} = \nabla_x (p \cdot \nabla_x) \frac{1}{4\pi |x + z|}.
\] (S.33)

Consequently, the integral (S.32) becomes

\[
A_2(x, t_x) = \frac{\mu_0}{4\pi} \nabla_x (p \cdot \nabla_x) \iiint_{\text{whole space}} d^3 z \frac{\delta'(t_x - |z|/c)}{|z|} \frac{1}{4\pi |x + z|}.
\] (S.34)

The remaining integral on the second line here looks like the LHS of the Lemma (6) for

\[
F(z) = \frac{1}{4\pi |x + z|},
\] (S.35)

hence

\[
\iiint_{\text{whole space}} d^3 z \frac{\delta'(t_x - |z|/c)}{|z|} \frac{1}{4\pi |x + z|} = c^2 \Theta(t) \left[ \left( 1 + \frac{r}{\partial r} \right) \int_0^{2\pi} d\Omega_n \frac{1}{4\pi |x + r n|} \right]_{r=ct}.
\] (S.36)

Note: in this formula \(r\) and \(n\) are the magnitude and the direction of \(z\) rather than \(x\).
Next, the angular integral in eq. (S.36) obtains from the Lemma (7):

$$
\oint d^2 \Omega_n \frac{1}{4\pi |\mathbf{x} + r\mathbf{n}|} = \frac{1}{\max(r, |\mathbf{x}|)} = \begin{cases} 
(1/r) & \text{for } r > |\mathbf{x}|, \\
(1/|\mathbf{x}|) & \text{for } r < |\mathbf{x}|.
\end{cases} \tag{S.37}
$$

Consequently,

for $r = ct > |\mathbf{x}|$,

$$
\left(1 + r \frac{\partial}{\partial r}\right) \oint d^2 \Omega_n \frac{1}{4\pi |\mathbf{x} + r\mathbf{n}|} = \left(1 + r \frac{\partial}{\partial r}\right) \frac{1}{r} = 0, \tag{S.38}
$$

for $r = ct < |\mathbf{x}|$,

$$
\left(1 + r \frac{\partial}{\partial r}\right) \oint d^2 \Omega_n \frac{1}{4\pi |\mathbf{x} + r\mathbf{n}|} = \left(1 + r \frac{\partial}{\partial r}\right) \frac{1}{|\mathbf{x}|} = \frac{1}{|\mathbf{x}|}, \tag{S.39}
$$

and therefore

$$
\int\int\int_{\text{whole space}} d^3 \mathbf{z} \frac{\delta'(t - |\mathbf{z}|/c)}{|\mathbf{z}|} \frac{1}{4\pi |\mathbf{x} + \mathbf{z}|} = c^2 \Theta(t) \Theta(|\mathbf{x}| - ct) \frac{1}{|\mathbf{x}|}. \tag{S.40}
$$

Plugging this formula back into eq. (S.34), we arrive at

$$
\mathbf{A}_2(\mathbf{x}, t) = \frac{c^2 \mu_0}{4\pi} \Theta(t) \nabla(|\mathbf{p} \cdot \nabla| \frac{\Theta(|\mathbf{x}| - ct)}{|\mathbf{x}|}) \tag{S.41}
$$

where $c^2 \mu_0 = 1/\varepsilon_0$.

For the later use in part (f), let me write down the entire vector potential for all times $t = t_x$,

$$
\mathbf{A}(\mathbf{x}, t) = \mathbf{A}_1(\mathbf{x}, t) + \mathbf{A}_2(\mathbf{x}, t) = \frac{\mu_0}{4\pi |\mathbf{x}|} \mathbf{p} \delta'(t - |\mathbf{x}|/c) + \frac{\Theta(t)}{4\pi \varepsilon_0} \nabla(|\mathbf{p} \cdot \nabla| \frac{\Theta(|\mathbf{x}| - ct)}{|\mathbf{x}|}). \tag{S.42}
$$

But for the current part (d) we assume $t < |\mathbf{x}|/c$, so the vector potential simplifies to

$$
\mathbf{A}(\mathbf{x}, t) = \frac{\Theta(t)}{4\pi \varepsilon_0} \nabla(|\mathbf{p} \cdot \nabla| \frac{1}{|\mathbf{x}|}) = \Theta(t) \frac{3(p \cdot \mathbf{n}_x)\mathbf{n}_x - \mathbf{p}}{4\pi \varepsilon_0 |\mathbf{x}|^3}. \tag{S.43}
$$
Problem 2(e):
At times $t < |\mathbf{x}|/c$ — before the light pulse from the dipole flash (6) reaches the point $\mathbf{x}$, — the vector potential (S.43) is a pure gradient of some scalar field, so its curl $\mathbf{B} = 0$. Thus, the magnetic field does not propagate faster than light.

As to the electric field
\[
\mathbf{E} = -\nabla \Phi - \frac{\partial}{\partial t} \mathbf{A},
\]
the vector potential (S.43) is a step function of time: It turns on at $t = 0$ and then stays constant until the light pulse of the dipole flash reaches the point $\mathbf{x}$. Consequently, at $t < |\mathbf{x}|/c$,
\[
\frac{\partial \mathbf{A}}{\partial t} = +\delta(t) \nabla (\mathbf{p} \cdot \nabla) \frac{1}{4\pi \varepsilon_0 |\mathbf{x}|}.
\]  

At the same time, in part (a) we found the scalar potential to be
\[
\Phi(\mathbf{x}, t) = -\delta(t) (\mathbf{p} \cdot \nabla) \frac{1}{4\pi \varepsilon_0 |\mathbf{x}|},
\]
hence
\[
\mathbf{E} = -\nabla \Phi - \frac{\partial}{\partial t} \mathbf{A} = 0.
\]
Thus, the superluminal terms in the scalar and the vector potentials cancel each other from the electric field! Consequently, the electric field — just like the magnetic field — does not propagate faster than light.

Problem 2(f):
In the solutions to part (d), I have written down eq. (S.42) for the vector potential at all times, both before the light front passes through the point in question and afterward. To simplify the notations in that formula, let me redefine $r = |\mathbf{x}|$ (instead of $r = |\mathbf{z}|$ we have used in part (d)), then eq. (S.42) becomes
\[
\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi r} \mathbf{p} \delta'(t - r/c) + \frac{\Theta(t)}{4\pi \varepsilon_0} \nabla (\mathbf{p} \cdot \nabla) \left( \frac{\Theta(r - ct)}{r} \right).
\]
From this formula we see that for $t > r/c$ — after the light front has moved on — the vector potential drops to zero, but there are all kinds of singular terms right when the light front
goes through at time $t = r/c$. To get all such terms, the derivative operators $\nabla(p \cdot \nabla)$ should act not only on the $1/r$ but also on the step function $\Theta(r - ct)$ in the numerator, thus

$$\nabla_i \left( \frac{\Theta(r - ct)}{r} \right) = n_i \left( \frac{\delta(r - ct)}{r} - \frac{\Theta(r - ct)}{r^2} \right), \quad (S.49)$$

$$\nabla_j \nabla_i \left( \frac{\Theta(r - ct)}{r} \right) = \delta_{ij} n_j n_i \left( \frac{\delta(r - ct)}{r} - \frac{\Theta(r - ct)}{r^2} \right)
+ n_j n_i \left( \frac{\delta'(r - ct)}{r} - \frac{2 \delta(r - ct)}{r^2} + \frac{2 \Theta(r - ct)}{r^3} \right)
= n_i n_j \frac{\delta'(r - ct)}{r}
- (3 n_i n_j - \delta_{ij}) \left( \frac{\delta(r - ct)}{r^2} - \frac{\Theta(r - ct)}{r^3} \right), \quad (S.50)$$

$$\nabla(p \cdot \nabla) \left( \frac{\Theta(r - ct)}{r} \right) = n \cdot p \frac{\delta'(r - ct)}{r}
- (3 n \cdot p - p) \left( \frac{\delta(r - ct)}{r^2} - \frac{\Theta(r - ct)}{r^3} \right). \quad (S.51)$$

At the same time,

$$\frac{\mu_0}{4\pi r} p \delta'(t - r/c) = -\frac{\mu_0 c^2}{4\pi r} p \delta'(r - ct) = -\frac{p}{4\pi \epsilon_0} \delta'(r - ct), \quad (S.52)$$

so putting all terms together, at the light front $r = ct$,

$$A(x, t) = \frac{1}{4\pi \epsilon_0} \left[ (n \cdot p - p) \frac{\delta'(r - ct)}{r} + (3 n \cdot p - p) \left( \frac{\delta(r - ct)}{r^2} - \frac{\Theta(r - ct)}{r^3} \right) \right]. \quad (S.53)$$

Problem 2(g):
We saw in part (e) that before the light front $E = B = 0$, and in part (f) we saw that after the light front $A = 0$ and hence also $E = B = 0$. Thus, the electric and the magnetic fields of the instant dipole flash exist only at the light front $r = ct$. To find them, we simply need to take one more space or time derivative of the vector potential (S.53).
In particular, the time derivative is rather simple:

\[-\frac{\partial}{\partial t} \Theta(r - ct) = c\delta(r - ct), \quad -\frac{\partial}{\partial t} \delta(r - ct) = c\delta'(r - ct), \quad -\frac{\partial}{\partial t} \delta'(r - ct) = c\delta''(r - ct),\]

hence the electric field is

\[E(x, t) = -\frac{\partial}{\partial t} A = \frac{c}{4\pi \varepsilon_0} \left[ \frac{(n \cdot p) - p}{r} \delta''(r - ct) \right.\]

\[\left. + \left( 3n \cdot (n \cdot p) - p \right) \frac{r}{r^2} - \frac{\delta(r - ct)}{r^3} \right] \]  

As to the magnetic field, we can simplify taking the curl of the vector potential (S.53) by noting that

\[\frac{n(n \cdot p) - p}{r} = \nabla (n \cdot p) \implies \nabla \times \frac{n(n \cdot p) - p}{r} = 0,\]

hence

\[\nabla \times \left( \frac{n(n \cdot p) - p}{r} \delta'(r - ct) \right) = \nabla \delta'(r - ct) \times \frac{n(n \cdot p) - p}{r} \]

\[= \frac{r}{r^2} \left( -n \cdot (n \cdot p) - p \right) \frac{n(n \cdot p) - p}{r} = \frac{p \times n}{r^2} \delta''(r - ct).\]

Likewise,

\[\frac{3n(n \cdot p) - p}{r^3} = \nabla \left( -\frac{n \cdot p}{r^2} \right) \implies \nabla \times \frac{3n(n \cdot p) - p}{r^3} = 0,\]

hence

\[\nabla \times \left( \frac{3n(n \cdot p) - p}{r^3} \left( r\delta(r - ct) - \Theta(r - ct) \right) \right) = \]

\[= \nabla \left( r\delta(r - ct) - \Theta(r - ct) \right) \times \frac{3n(n \cdot p) - p}{r^3} \]

\[= \left( r\delta'(r - ct) \right) n \times \frac{3n(n \cdot p) - p}{r^3} \]

\[= \frac{p \times n}{r^2} \delta'(r - ct).\]

Altogether,

\[B = \nabla \times A \text{[from eq. (S.53)]} = \frac{p \times n}{4\pi \varepsilon_0} \left[ \frac{\delta''(r - ct)}{r} + \frac{\delta'(r - ct)}{r^2} \right].\]