Problem 1(a):
The quadrupole moment tensor of a system of point charges is

\[ Q_{ij} = \sum_n q_n \left( \frac{3}{2} x_{n,i} x_{n,j} - \frac{1}{2} r_n^2 \right). \quad (S.1) \]

The 4 charges in question are all in the same plane — which we take to be the \((x, y)\) plane, — hence \(Q_{xz} = Q_{yz} = 0\). Also, all 4 charges lie at the same distance \(r = a/\sqrt{2}\) from the origin and the net charge \(\sum_n q_n\) vanishes, hence \(\sum_n q_n r_n^2 = 0\) and therefore

\[ Q_{zz} = 0 \quad \text{and} \quad Q_{xx} + Q_{yy} = 0. \quad (S.2) \]

The remaining independent components of the quadrupole tensor form a complex combination

\[ Q = Q_{xx} - Q_{yy} + 2iQ_{xy} = \frac{3}{2} \sum_n q_n (x_n + iy_n)^2. \quad (S.3) \]

For the charges at the corners of a rotating square

\[ \forall n:\ \ q_n (x_n + iy_n)^2 = +q \frac{a^2}{2} \times e^{2i\omega t} \quad (S.5) \]
and hence
\[ Q = 3qa^2 \times e^{2i\omega t}. \] (S.6)

In terms of the quadrupole tensor components, this means
\[ Q_{xx} = -Q_{yy} = \frac{1}{2} \text{Re}(Q) = \frac{3}{2}qa^2 \times \cos(2\omega t), \quad Q_{xy} = \frac{1}{2} \text{Im}(Q) = \frac{3}{2}qa^2 \times \sin(2\omega t), \] (S.7)
or in matrix notations
\[ Q_{ij} = \frac{3qa^2}{2} \begin{pmatrix} \cos(2\omega t) & \sin(2\omega t) & 0 \\ \sin(2\omega t) & -\cos(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \] (S.8)

Note that this quadrupole tensor oscillates with frequency \(2\omega\), i.e., twice the rotation frequency of the charges.

**Problem 1 (b–c):**
As explained in class, the EM power radiated in a particular direction \(\mathbf{n}\) is
\[ \frac{dP}{d\Omega} = \frac{Z_0\omega_{osc}^2}{2c^2} \times (|\mathbf{f}(\mathbf{n})|^2 - |\mathbf{n} \cdot \mathbf{f}(\mathbf{n})|^2) \] (S.9)

where
\[ \mathbf{f}(\mathbf{n}) = \frac{1}{4\pi} \iiint d^3y \mathbf{J}(y) \exp(-i\mathbf{n} \cdot \mathbf{y}). \] (S.10)

In the long wavelength approximation, the leading contribution to the \(\mathbf{f}\) comes from the lowest oscillating multipole moment, electric or magnetic. For the system at hand, the lowest oscillating moment is the electric quadrupole; in light of eq. (S.8), it has frequency \(\omega_{osc} = 2\omega\) and amplitude
\[ Q_{ij} = \frac{3qa^2}{2} \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \] (S.11)
For a general electric quadrupole,

\[ f_j(n) = \frac{\omega_{\text{osc}}^2}{12\pi c} Q_{jk} n_k, \quad \text{(S.12)} \]

so for the quadrupole in question

\[
\begin{pmatrix}
  f_x \\
  f_y \\
  f_z 
\end{pmatrix} = \frac{\omega_{\text{osc}}^2 q a^2}{8\pi c} \begin{pmatrix}
  1 \\
  i \\
  0
\end{pmatrix} (n_x + in_y), \quad \text{(S.13)}
\]

or in spherical coordinates

\[
\begin{pmatrix}
  f_x \\
  f_y \\
  f_z 
\end{pmatrix} = \frac{\omega_{\text{osc}}^2 q a^2}{8\pi c} \begin{pmatrix}
  1 \\
  i \\
  0
\end{pmatrix} \sin \theta e^{i\phi}. \quad \text{(S.14)}
\]

Consequently,

\[
f^* \cdot f = \frac{\omega_{\text{osc}}^4 q^2 a^4}{64\pi^2 c^2} \times 2 \sin^2 \theta, \quad \text{(S.15)}
\]

\[
n \cdot f = \frac{\omega_{\text{osc}}^4 q^2 a^4}{64\pi^2 c^2} \times (\sin \theta e^{i\phi})^2, \quad \text{(S.16)}
\]

hence

\[
(|f(n)|^2 - |n \cdot f(n)|^2) = \frac{\omega_{\text{osc}}^4 q^2 a^4}{64\pi^2 c^2} \times (2 \sin^2 \theta - \sin^4 \theta), \quad \text{(S.17)}
\]

and therefore

\[
\frac{dP}{d\Omega} = \frac{Z_0 q^2 a^4 \omega_{\text{osc}}^6}{128\pi^2 c^4} \times \sin^2 \theta (2 - \sin^2 \theta). \quad \text{(S.18)}
\]

In particular, the angular dependence of the radiated power has form

\[
\frac{dP}{d\Omega} \propto \sin^2 \theta (2 - \sin^2 \theta) = 1 - \cos^4 \theta. \quad \text{(S.19)}
\]
Graphically,

As to the total power radiated by the rotating quadrupole,

\[
P_{\text{net}} = \frac{Z_0 q^2 a^4 \omega_{\text{osc}}^6}{128 \pi^2 c^4} \times \int \int d^2 \Omega (1 - \cos^4 \theta)
\]

where \( \omega_{\text{osc}} = 2\omega \) and

\[
\int \int d^2 \Omega (1 - \cos^4 \theta) = 2 \pi \int_{-1}^{+1} d \cos \theta (1 - \cos^4 \theta) = 4 \pi \times \left( 1 - \frac{1}{5} \right) = \frac{16 \pi}{5}.
\]

Thus altogether,

\[
P_{\text{net}} = \frac{8 Z_0 q^2 a^4 \omega^6}{5 \pi c^4} = \frac{8 q^2}{5 \pi \varepsilon_0} \times \frac{a^4 \omega^6}{c^5}.
\]
Problem 2(a):
The retarded Green’s function of the D’Alembertian operator is
\[ G_R(\mathbf{x} - \mathbf{y}, t_x - t_y) = \frac{\delta(t_x - t_y - |\mathbf{x} - \mathbf{y}|/c)}{4\pi|\mathbf{x} - \mathbf{y}|}, \quad (S.23) \]
hence in the Landau gauge for the EM potentials \( \Phi \) and \( \mathbf{A} \),
\begin{align*}
\Phi(\mathbf{x}, t) &= \frac{1}{4\pi\epsilon_0} \iiint d^3y \frac{\rho(y, t - |\mathbf{x} - \mathbf{y}|/c)}{|\mathbf{x} - \mathbf{y}|}, \\
\mathbf{A}(\mathbf{x}, t) &= \frac{\mu_0}{4\pi} \iiint d^3y \frac{\mathbf{J}(y, t - |\mathbf{x} - \mathbf{y}|/c)}{|\mathbf{x} - \mathbf{y}|}. \quad (S.24)
\end{align*}
In particular, for the current \( \mathbf{J} \) as in eq. (1), the vector potential is
\[ \mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \iiint d^3y \frac{\mathbf{d}(t - |\mathbf{x} - \mathbf{y}|/c)}{|\mathbf{x} - \mathbf{y}|} \delta^{(3)}(\mathbf{y}) = \frac{\mu_0}{4\pi} \frac{\mathbf{d}(t - |\mathbf{x}|/c)}{|\mathbf{x}|}, \quad (S.25) \]
in perfect agreement with eq. (2). Note: in my notations, \( \dot{\mathbf{d}} \) is the time derivative of the dipole moment \( \mathbf{d} \).

For the scalar potential, the calculation is more involved:
\begin{align*}
\Phi(\mathbf{x}, t) &= -\frac{1}{4\pi\epsilon_0} \iiint d^3y \frac{\mathbf{d}(t - |\mathbf{x} - \mathbf{y}|)}{|\mathbf{x} - \mathbf{y}|} \cdot \nabla \delta^{(3)}(\mathbf{y}) \\
&= +\frac{1}{4\pi\epsilon_0} \left[ \nabla_y \cdot \frac{\mathbf{d}(t - |\mathbf{x} - \mathbf{y}|)}{|\mathbf{x} - \mathbf{y}|} \right] \bigg|_{\mathbf{y} = 0} \\
&= -\frac{1}{4\pi\epsilon_0} \left[ \nabla_x \cdot \frac{\mathbf{d}(t - |\mathbf{x} - \mathbf{y}|)}{|\mathbf{x} - \mathbf{y}|} \right] \bigg|_{\mathbf{y} = 0} \\
&= -\frac{1}{4\pi\epsilon_0} \nabla_x \cdot \frac{\mathbf{d}(t - |\mathbf{x}|/c)}{|\mathbf{x}|}. \quad (S.26)
\end{align*}
On the last line here, the space derivative \( \nabla_x \) acts on both the denominator and on the numerator \( \mathbf{d}(t_{\text{ret}}) \) since the retarded time depends on \( \mathbf{x} \),
\[ \nabla(t_{\text{ret}} = t - r/c) = -\frac{\mathbf{n}}{c}, \quad (S.27) \]
hence
\[ \nabla \cdot \mathbf{d}(t_{\text{ret}}) = -\frac{n}{c} \cdot \dot{\mathbf{d}}(t_{\text{ret}}) \] (S.28)

and therefore
\[ \Phi(\mathbf{x}, t) = + \frac{1}{4\pi\epsilon_0} \left[ \frac{n}{rc} \cdot \dot{\mathbf{d}} + \frac{n}{r^2} \cdot \dot{\mathbf{d}} \right]_{\text{ret}}, \] (S.29)

exactly as in eq. (2).

**Problem 2(b):**
Let’s start with the magnetic field

\[ \mathbf{B}(\mathbf{x}, t) = \nabla \times \mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \nabla \times \left( \frac{\dot{\mathbf{d}}(t_{\text{ret}})}{r} \right). \] (S.30)

Similar to eq. (S.28),
\[ \nabla \times \dot{\mathbf{d}}(t_{\text{ret}}) = -\frac{n}{c} \times \ddot{\mathbf{d}}(t_{\text{ret}}), \] (S.31)

hence
\[ \nabla \times \left( \frac{\dot{\mathbf{d}}(t_{\text{ret}})}{r} \right) = -\frac{n}{rc} \times \ddot{\mathbf{d}}(t_{\text{ret}}) - \frac{n}{r^2} \times \dot{\mathbf{d}}(t_{\text{ret}}) \] (S.32)

and therefore
\[ \mathbf{B}(\mathbf{x}, t) = -\frac{\mu_0}{4\pi} \left[ \frac{n}{rc} \times \dot{\mathbf{d}} + \frac{n}{r^2} \times \ddot{\mathbf{d}} \right]_{\text{ret}}. \] (S.33)

In a similar manner, taking the gradient of the scalar potential \( \Phi(\mathbf{x}, t) \), we find
\[ \nabla_i \left( \frac{n}{rc} \cdot \dot{\mathbf{d}}(t_{\text{ret}}) \right) = \left( \nabla_i \left( \frac{n_j}{rc} \right) \right) \cdot \dot{\mathbf{d}}_j(t_{\text{ret}}) + \frac{n_j}{rc} \left( \nabla_i \dot{d}_j(t_{\text{ret}}) \right) \]
\[ = \delta_{ij} - \frac{2n_i n_j}{r^2 c} \dot{d}_j(t_{\text{ret}}) - \frac{n_j}{rc} \frac{n_i}{c} \ddot{d}_j(t_{\text{ret}}), \] (S.34)
\[ \nabla_i \left( \frac{n_i}{r^2} \cdot d(t_{\text{ret}}) \right) = \left( \nabla_i \left( \frac{n_j}{r^2} \right) \right) d_j(t_{\text{ret}}) + \frac{n_j}{r^2} \left( \nabla_i d_j(t_{\text{ret}}) \right) \]
\[ = \frac{\delta_{ij} - 3n_in_j}{r^3} d_j(t_{\text{ret}}) - \frac{n_j}{r^2 c} \frac{n_i}{c} d_j(t_{\text{ret}}), \quad (S.35) \]

and therefore
\[ \nabla_i \Phi(x, t) = \frac{1}{4\pi\epsilon_0} \left[ \frac{\delta_{ij} - 3n_in_j}{r^3} d_j + \frac{\delta_{ij} - 3n_in_j}{r^2c} \frac{\dot{d}_j}{c} - \frac{n_i n_j}{rc^2} \frac{\ddot{d}_j}{c} \right]_{\text{ret}}. \quad (S.36) \]

At the same time,
\[ \frac{\partial}{\partial t} A_i(x, t) = \frac{\mu_0}{4\pi r} \frac{\ddot{d}_i(t_{\text{ret}})}{r}, \quad (S.37) \]

thus
\[ E_i(x, t) = -\nabla_i \Phi(x, t) - \frac{\partial A_i}{\partial t} \]
\[ = -\frac{1}{4\pi\epsilon_0} \left[ \frac{\delta_{ij} - 3n_in_j}{r^3} d_j + \frac{\delta_{ij} - 3n_in_j}{r^2c} \frac{\dot{d}_j}{c} + \frac{\delta_{ij} - n_in_j}{rc^2} \frac{\ddot{d}_j}{c} \right]_{\text{ret}}. \quad (S.38) \]

Or in vector notations,
\[ E(x, t) = -\frac{1}{4\pi\epsilon_0} \left[ \frac{d - 3n(n \cdot d)}{r^3} + \frac{\dot{d} - 3n(n \cdot \dot{d})}{r^2c} + \frac{\ddot{d} - n(n \cdot \ddot{d})}{rc^2} \right]_{\text{ret}}. \quad (S.39) \]

**Problem 2(c):**
For a harmonically oscillating dipole moment \( d(t) = d \exp(-i\omega t) \), we have
\[ d(t_{\text{ret}}) = d \exp(ikr - i\omega t), \]
\[ \frac{1}{c} \dot{d}(t_{\text{ret}}) = -ikd \exp(ikr - i\omega t), \]
\[ \frac{1}{c^2} \ddot{d}(t_{\text{ret}}) = -k^2d \exp(ikr - i\omega t). \quad (S.40) \]

Consequently, eq. (S.33) for the magnetic field becomes
\[ B(x, t) = -\frac{\mu_0 c}{4\pi} e^{ikr - i\omega t} \left[ \frac{-k^2 n \times d}{r} + \frac{-ikn \times d}{r^2} \right] \]
\[ = + \frac{k^2}{4\pi\epsilon_0 c} e^{ikr - i\omega t} \left( 1 + \frac{i}{kr} \right) (n \times d), \quad (S.41) \]
exactly as in eq. (3.a). Likewise, eq. (S.39) for the electric field becomes

$$ E(x, t) = -\frac{1}{4\pi\epsilon_0} e^{ikr-i\omega t} \left[ \frac{d - 3n(n \cdot d)}{r^3} - ik \frac{d - 3n(n \cdot d)}{r^2} - k^2 \frac{d - n(n \cdot d)}{r} \right] $$

$$ = + \frac{k^2}{4\pi\epsilon_0} \frac{e^{ikr-i\omega t}}{r} \left[ \frac{i}{kr} \left( 1 + \frac{i}{kr} \right) (d - 3n(n \cdot d)) - n \times (n \times d) \right], \quad (S.42) $$

in perfect agreement with eq. (3.b).

**Problem 2(d):**

Eqs. (3) for the magnetic and the electric fields apply for all distances $r$ from the dipole — short, medium, and long — as long as the dipole itself may be approximated as point-like, So let’s take a closer look at their long-distance and short-distance limits, where the distances are viewed as long or short by comparison with the wavelength $\lambda = 2\pi/k$.

In the long distance regime $r \gg \lambda$, we may neglect all the negative powers of $kr$ in eqs. (3), which leaves us with

$$ eB(x, t) \approx -\frac{k^2}{4\pi\epsilon_0} \frac{e^{ikr-i\omega t}}{r} (n \times d), \quad (S.43) $$

$$ E(x, t) \approx -\frac{k^2}{4\pi\epsilon_0} \frac{e^{ikr-i\omega t}}{r} (n \times (n \times d)). \quad (S.44) $$

These are precisely the radiation fields of a harmonic dipole we have discussed in class. Note that they diminish with distance as $1/r$, so that the radiation power density spreads out as $1/r^2$.

On the other hand, in the short distance regime $r \ll \lambda$, we focus on the highest negative powers of $kr$ in eqs. (3), and we may also approximate $\exp(ikr) \approx 1$. Consequently, the short-distance limit of the electric field is

$$ E(x, t) \approx -\frac{k^2}{4\pi\epsilon_0} \frac{e^{-i\omega t}}{r} \frac{-1}{k^2 r^2} (d - n(n \cdot d)) $$

$$ = + \frac{1}{4\pi\epsilon_0} \frac{d - 3n(n \cdot d)}{r^3} e^{-i\omega t}. \quad (S.45) $$

This is a quasistatic Coulomb field of the electric dipole $d(t) = de^{-i\omega t}$. That is, at any given instance of time $t$, the field (S.44) is the Coulomb field of the dipole moment we happen to have at time $t$. As any good dipole field, it scales with distance as $1/r^3$. 

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As to the magnetic field in the short-distance regime, the leading term in eq. (3.a) is

\[ B(x, t) = -\frac{i\mu_0\omega}{4\pi} \frac{n \times d}{r^2} e^{-i\omega t}. \]  

(S.46)

Unlike the electric field (S.45), the short-distance magnetic field scales with distance as \(1/r^2\), slower than any quasistatic magnetic multipole, but faster than the \(1/r\) radiation-zone fields. Also, the magnetic field (S.44) is not a quasistatic field, since it vanishes for \(\omega \to 0\). Instead, this magnetic field is induced by the displacement current due to the time-dependent dipole field (S.39) in the short-distance zone. Indeed,

\[ \nabla \times H (\text{from eq. (S.46)}) = -\frac{i\omega}{4\pi} \nabla \times \left( \frac{n \times d}{r^2} \right) e^{-i\omega t} \]

\[ = -\frac{i\omega}{4\pi} \frac{d - 3n(n \cdot d)}{r^3} e^{-i\omega t} \]  

(S.47)

\[ = \frac{\partial}{\partial t} \left( D = \epsilon_0 E (\text{from eq. (S.45)}) \right). \]

Problem 3(a):

In part (b) of problem (2) we saw that for general \(d(t)\) the electric and the magnetic fields emitted by the dipole are

\[ B(x, t) = -\frac{\mu_0}{4\pi} \left[ \frac{n}{rc} \times \ddot{d} + \frac{n}{r^2} \times \dot{d} \right]_{\text{ret}}, \]  

(S.33)

\[ E(x, t) = -\frac{1}{4\pi\epsilon_0} \left[ \frac{d - 3n(n \cdot d)}{r^3} + \frac{\ddot{d} - 3n(n \cdot \dot{d})}{r^2c} + \frac{\ddot{d} - n(n \cdot \ddot{d})}{rc^2} \right]_{\text{ret}}. \]  

(S.39)

In the long distance limit, both fields may be approximated by the terms which behave as \(1/r\) rather than \(1/r^2\) or \(1/r^3\), thus

\[ B(x, t) \approx -\frac{\mu_0}{4\pi c} \frac{n}{r} \times \ddot{d}(t_{\text{ret}}), \]  

(S.48)

\[ E(x, t) \approx -\frac{1}{4\pi\epsilon_0 c^2} \frac{[\ddot{d} - n(n \cdot \dot{d})](t_{\text{ret}})}{r} \approx -cn \times B(x, t). \]  

(S.49)
Consequently, the Poynting vector is

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} = -\frac{c}{\mu_0} (\mathbf{n} \times \mathbf{B}) \times \mathbf{B} = +\frac{c}{\mu_0} \left( \mathbf{n} (\mathbf{B}^2) - \mathbf{B} (\mathbf{n} \cdot \mathbf{B}) \right) = +\frac{c \mathbf{B}^2}{\mu_0} \mathbf{n} \tag{S.50}$$

where the last equality here follows from $\mathbf{B} \propto \mathbf{n} \times \ddot{\mathbf{d}} \implies \mathbf{n} \cdot \mathbf{B} = 0$. Specifically, for the magnetic field as in eq. (S.48), the Poynting vector is

$$\mathbf{S} = \frac{\mu_0}{16\pi^2c} [\mathbf{n} \times \dddot{\mathbf{d}}(t_{\text{ret}})]^2 \frac{\mathbf{n}}{r^2}. \tag{S.51}$$

Note that this energy flux is directed radially outward and diminishes with distance as $1/r^2$. Consequently, the power radiated per unit of solid angle in the direction $\mathbf{n}$ is

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2c} [\mathbf{n} \times \dddot{\mathbf{d}}(t_{\text{ret}})]^2, \tag{S.52}$$

and the net power radiated by the dipole is

$$\int \frac{dP}{d\Omega} d\Omega = \frac{\mu_0}{6\pi c} \dddot{d}(t_{\text{ret}}) = \frac{Z_0}{6\pi c^2} \dddot{d}(t_{\text{ret}}). \tag{4}$$

BTW, the retarded time $t_{\text{ret}} = t - r/c$ is retarded relative to the time $t$ at which we detect this radiation at long distance $r$ from the dipole. By the clock of the dipole itself, the energy loss happens at the same time as the $\dddot{d}$, thus

$$\frac{dU_{\text{dipole}}(t')}{dt'} = -\frac{Z_0}{6\pi c^2} \dddot{d}(t'). \tag{S.53}$$
Problem 3(b):
The parallel-plate capacitor in question has capacitance
\[ C = \frac{\epsilon_0 A}{b}. \] (S.54)

When it’s charged to initial charge \( Q_0 \) and then allowed to discharge via resistor \( R \), it’s charge decreases exponentially as
\[ Q(t) = Q_0 \times \exp(-t/\tau) \quad \text{for} \quad \tau = RC. \] (S.55)

The dipole moment of this capacitor is
\[ d(t) = bQ(t) = bQ_0 \exp(-t/\tau), \] (S.56)
hence
\[ \ddot{d} = \frac{bQ_0}{\tau^2} \exp(-t/\tau), \] (S.57)
which causes EM radiation at net power
\[ P = \frac{Z_0}{6\pi c^2} \frac{b^2 Q_0^2}{\tau^4} \times \exp(-2t/\tau). \] (S.58)

Integrating this power over the discharge time, we find the net energy carried by the EM radiation to be
\[ \Delta U_{\text{EM}} = \int_0^\infty dt P(t) = \frac{Z_0}{6\pi c^2} \frac{b^2 Q_0^2}{\tau^4} \times \int_0^\infty dt e^{-2t/\tau} = \frac{Z_0}{6\pi c^2} \frac{b^2 Q_0^2}{\tau^4} \times \frac{\tau}{2}. \] (S.59)

Compared to the initial energy stored in the capacitor
\[ U_0 = \frac{Q_0^2}{2C} = \frac{Q_0^2 b}{2\epsilon_0 A}, \] (S.60)
the fraction of this energy carried by the EM radiation is
\[ \frac{\Delta U_{\text{EM}}}{U_0} = \frac{Z_0 \epsilon_0}{6\pi c^2} \times \frac{AB}{\tau^3} = \frac{1}{6\pi} \times \frac{AB}{(c\tau)^3} \] (S.61)
where the second equality follows from \( Z_0 \epsilon_0 c = 1 \).
Problem 3(b):
For the specific example of $A = 100 \text{ cm}^2 = 0.01 \text{ m}^2$, $d = 1 \text{ mm} = 10^{-3} \text{ m}$ and $R = 10 \Omega$, we have

$$C = \frac{\varepsilon_0 A}{b} = 88.5 \text{ pF}, \quad \tau = RC = 0.885 \text{ ns}, \quad c\tau = 0.265 \text{ m}, \quad (S.62)$$

and hence

$$\frac{\Delta U_{\text{EM}}}{U_0} = \frac{1}{6\pi} \times \frac{AB}{(c\tau)^3} = \frac{10^{-5} \text{ m}^3}{6\pi (0.265 \text{ m})^3} = 2.85 \times 10^{-5}. \quad (S.63)$$