Preliminaries for Problem 2

As explained in the textbook §9.7, the electric multipole field have form

\[ H^{(E)}_{\ell,m}(r, \theta, \phi) = f_\ell(kr) L Y_{\ell,m}(\theta, \phi), \]
\[ E^{(E)}_{\ell,m}(r, \theta, \phi) = \frac{iZ_0}{k} \nabla \times H^{(E)}_{\ell,m}(r, \theta, \phi), \]  

while the magnetic multipole fields have form

\[ E^{(M)}_{\ell,m}(r, \theta, \phi) = Z_0 g_\ell(kr) L Y_{\ell,m}(\theta, \phi), \]
\[ H^{(M)}_{\ell,m}(r, \theta, \phi) = -\frac{i}{kZ_0} \nabla \times E^{(E)}_{\ell,m}(r, \theta, \phi). \]  

In these formulae, \( Y_{\ell,m}(\theta, \phi) \) are spherical harmonics, \( L \) is the angular momentum operator (without the \( \hbar \) factor),

\[ L = -i \mathbf{x} \times \nabla, \]  

and \( f_\ell(kr) \) and \( g_\ell(kr) \) are some linear combinations of the spherical Bessel functions \( j_\ell(kr) \) and \( n_\ell(kr) \). As explained in the textbook §9.6,

\[ j_\ell(x) = (-x)^\ell \left( \frac{d}{dx} \right)^\ell \frac{\sin(x)}{x} \xrightarrow{x \to \infty} \frac{\sin(x - \ell \pi/2)}{x}, \]
\[ n_\ell(x) = -(-x)^\ell \left( \frac{d}{dx} \right)^\ell \frac{\cos(x)}{x} \xrightarrow{x \to \infty} \frac{\cos(x - \ell \pi/2)}{x}, \]  

so to get a divergent spherical wave which behaves as

\[ f_\ell, g_\ell \propto \frac{\exp(+ikr)}{r}, \]  

we should combine the \( j_\ell \) and the \( n_\ell \) functions into the spherical Hankel functions of the
first kind

\[ h^{(1)}_\ell(x) = j_\ell(x) + i n_\ell(x), \quad (S.6) \]

or rather

\[ i^{\ell+1} h^{(1)}_\ell(x) = -(-ix)^\ell \left( \frac{1}{x} \frac{d}{dx} \right)^\ell e^{+ix} \xrightarrow{x \to \infty} \frac{e^{+ix}}{x}. \quad (S.7) \]

Thus, we let

\[ f_\ell(kr) = g_\ell(kr) = i^{\ell+1} k \times h^{(1)}_\ell(kr), \quad (S.8) \]

up to some overall amplitude factors. In particular, for \( \ell = 0, 1, 2 \) we have

\[
\begin{align*}
g_0(kr) &= \frac{e^{ikr}}{r}, \\
g_1(kr) &= \frac{e^{ikr}}{r} \left( 1 + \frac{i}{kr} \right), \\
g_2(kr) &= \frac{e^{ikr}}{r} \left( 1 + \frac{3i}{kr} - \frac{3}{(kr)^2} \right). \quad (S.9)
\end{align*}
\]

Next, consider the angular profiles

\[ X_{\ell,m}(\theta, \phi) = \mathbf{L} Y_{\ell,m}(\theta, \phi). \quad (S.10) \]

The action of the angular momentum operator \( \mathbf{L} \) on the spherical harmonics \( Y_{\ell,m}(\theta, \phi) \) should be painfully familiar to you from the undergraduate quantum mechanics class:

\[
\begin{align*}
L_x Y_{\ell,m} &= \frac{1}{2} \sqrt{(\ell - m)(\ell + 1 + m)} \times Y_{\ell,m+1} + \frac{1}{2} \sqrt{(\ell + m)(\ell + 1 - m)} \times Y_{\ell,m-1}, \\
L_y Y_{\ell,m} &= -\frac{i}{2} \sqrt{(\ell - m)(\ell + 1 + m)} \times Y_{\ell,m+1} + \frac{i}{2} \sqrt{(\ell + m)(\ell + 1 - m)} \times Y_{\ell,m-1}, \\
L_z Y_{\ell,m} &= m \times Y_{\ell,m}. \quad (S.11)
\end{align*}
\]

so evaluating all the vector spherical harmonics (S.10) for \( \ell = 1 \) and \( \ell = 2 \) is a straightforward albeit rather tedious exercise.
However, there is a more efficient way of handling the dipoles and the quadrupoles in terms of the dipole moment vector or the quadrupole moment tensor. Indeed, for $\ell = 1$

$$Y_{1,m}(\theta, \phi) = n(\theta, \phi) \cdot d_m$$  \hspace{1cm} (S.12)

for an appropriate dipole moment vector $d_m$, namely

$$d_0 = \sqrt{\frac{3}{4\pi}} (0, 0, 1), \quad d_{\pm 1} = \sqrt{\frac{3}{8\pi}} (\mp 1, -i, 0).$$  \hspace{1cm} (S.13)

Therefore,

$$X_{1,m} = -ix \times \nabla (n \cdot d_m) = -ir n \times \frac{d_m - n(n \cdot d_m)}{r} = -in \times d_m.$$  \hspace{1cm} (S.14)

Likewise, for $\ell = 2$

$$Y_{2,m}(n) = n \cdot Q_m \cdot n$$  \hspace{1cm} (S.15)

for an appropriate quadrupole moment tensor $Q_m$; in matrix form

$$Q_0 = \sqrt{\frac{5}{16\pi}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +2 \end{pmatrix},$$

$$Q_{\pm 1} = \sqrt{\frac{15}{32\pi}} \begin{pmatrix} 0 & 0 & \mp 1 \\ 0 & 0 & -i \\ \mp 1 & -i & 0 \end{pmatrix},$$  \hspace{1cm} (S.16)

$$Q_{\pm 2} = \sqrt{\frac{15}{32\pi}} \begin{pmatrix} +1 & \pm i & 0 \\ \pm i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For such harmonics

$$X_{2,m} = -ix \times \nabla (n \cdot Q_m \cdot n) = -ir n \times \frac{2Q_m \cdot n - 2n(n \cdot Q_m \cdot n)}{r} = -2in \times (Q_m \cdot n).$$  \hspace{1cm} (S.17)
Solutions for Problem 2

Given the above preliminaries, we may now write down explicit formulae for the dipole and the quadrupole fields.

(a) Electric dipole fields:
According to eqs. (S.1), (S.9), and (S.14), the magnetic field is

\[ H_{1,m}^{(E)}(x) = f_1(kr) L Y_{1,m}(n) = \frac{e^{ikr}}{r} \left( 1 + \frac{i}{kr} \right) (-i n \times d_m), \]  
(S.18)

while the electric field follows as

\[ E_{1,m}^{(E)}(x) = \frac{im_0}{k} \nabla \times H_{1,m}^{(E)}(x). \]  
(S.19)

To take the curl here, we start with derivatives of the radial and the angular profiles by themselves,

\[ \nabla \left[ \frac{e^{ikr}}{r} \left( 1 + \frac{1}{kr} \right) \right] = \frac{e^{ikr}}{r} \left( 1 + \frac{2i}{kr} - \frac{2}{(kr)^2} \right) i k n, \]  
(S.20)

\[ \nabla \times (-i n \times d_m) = i \frac{d_m + n(n \cdot d)}{r}. \]  
(S.21)

Consequently,

\[ E_{1,m}^{(E)}(x) = i m_0 \frac{e^{ikr}}{r} \left( 1 + \frac{2i}{kr} - \frac{2}{(kr)^2} \right) (n \times n \times d_m = n(n \cdot d_m) - d_m) \]

\[ + i m_0 \frac{e^{ikr}}{r} \left( 1 + \frac{i}{kr} \right) \frac{i}{kr} (d_m + n(n \cdot d)) \]

\[ = i m_0 \frac{e^{ikr}}{r} \left[ \left( 1 + \frac{3i}{kr} - \frac{3}{(kr)^2} \right) (n \cdot d_m)n - \left( 1 + \frac{i}{kr} - \frac{1}{(kr)^2} \right) d_m \right]. \]  
(S.22)
(b) Magnetic dipole fields:

Proceeding exactly as in part (a) but using eqs. (S.2) instead of (S.1), we have the electric field

\[
E_{1,m}^{(M)}(x) = Z_0 g_1(kr) \mathbf{L} Y_{1,m}(\mathbf{n}) = Z_0 \frac{e^{ikr}}{r} \left( 1 + \frac{i}{kr} \right) (-i \mathbf{n} \times \mathbf{d}_m),
\]

while the magnetic field follows as

\[
H_{1,m}^{(M)}(x) = -\frac{i}{kZ_0} \nabla E_{1,m}^{(M)}(x).
\]

Taking the derivatives exactly as in part (a), we obtain

\[
H_{1,m}^{(M)}(x) = -i \frac{e^{-kr}}{r} \left[ \left( 1 + \frac{3i}{kr} - \frac{3}{(kr)^2} \right) (\mathbf{n} \cdot \mathbf{d}_m) \mathbf{n} - \left( 1 + \frac{i}{kr} - \frac{1}{(kr)^2} \right) \mathbf{d}_m \right].
\]

(c) Electric quadrupole fields:

This time, we combine eqs. (S.1), (S.9), and (S.17) to get the magnetic field

\[
H_{2,m}^{(E)}(x) = f_2(kr) \mathbf{L} Y_{2,m}(\mathbf{n}) = \frac{e^{ikr}}{r} \left( 1 + \frac{3i}{kr} - \frac{3}{(kr)^2} \right) (-i \mathbf{n} \times (Q_m \cdot \mathbf{n})),
\]

while the electric field follows as

\[
E_{2,m}^{(E)}(x) = \frac{iZ_0}{k} \nabla \times H_{2,m}^{(E)}(x).
\]

To take the curl here, we proceed as in part (a) and start with the derivatives of the radial and the angular profiles by themselves:

\[
\nabla \left[ \frac{e^{ikr}}{r} \left( 1 + \frac{3i}{kr} - \frac{3}{(kr)^2} \right) \right] = \frac{e^{ikr}}{r} \left( 1 + \frac{4i}{kr} - \frac{9}{(kr)^2} - \frac{9i}{(kr)^3} \right) i \mathbf{n},
\]

\[
\nabla \times (-i \mathbf{n} \times (Q_m \cdot \mathbf{n})) = \frac{i}{r} (Q_m \cdot \mathbf{n} + 2 \mathbf{n} \cdot (Q_m \cdot \mathbf{n})).
\]
Hence altogether,

\[
E^{(E)}_{2,m}(x) = iZ_0 \frac{e^{ikr}}{r} \left( 1 + \frac{4i}{kr} - \frac{9}{(kr)^2} - \frac{9i}{(kr)^3} \right) (n \times (n \times (Q_m \cdot n))) \\
- \frac{Z_0}{kr} \frac{e^{ikr}}{r} \left( 1 + \frac{3i}{kr} - \frac{3}{(kr)^2} \right) (Q_m \cdot n + 2n(Q_m \cdot n)) \\
= iZ_0 \frac{e^{ikr}}{r} \left[ \left( 1 + \frac{4i}{kr} - \frac{9}{(kr)^2} - \frac{9i}{(kr)^3} \right) (-Q_m \cdot n + n(Q_m \cdot n)) \\
+ \left( \frac{i}{rk} - \frac{3}{(kr)^2} - \frac{3i}{(kr)^3} \right) (Q_m \cdot n + 2n(Q_m \cdot n)) \right] \\
= iZ_0 \frac{e^{ikr}}{r} \left[ \left( 1 + \frac{6i}{kr} - \frac{15}{(kr)^2} - \frac{15i}{(kr)^3} \right) (n \cdot Q_m \cdot n) \right] \\
- \left( 1 + \frac{3i}{kr} - \frac{6}{(kr)^2} - \frac{6i}{(kr)^3} \right) Q_m \cdot n \right].
\]  

(S.30)

(d) Magnetic quadrupole fields:

Proceeding exactly as in part (c) but starting with eq. (S.2) instead of eq. (S.1), we have the electric field

\[
E^{(E)}_{2,m}(x) = Z_0 g_2(kr) \mathbf{L} Y_{2,m}(n) = Z_0 \frac{e^{ikr}}{r} \left( 1 + \frac{3i}{kr} - \frac{3}{(kr)^2} \right) (-in \times (Q_m \cdot n)),
\]  

(S.31)

while the magnetic field follows as

\[
H^{(E)}_{2,m}(x) = \frac{-i}{Z_0 k} \nabla \times H^{(E)}_{2,m}(x).
\]  

(S.32)

Taking the curl here works exactly as in part (c), and we end up with

\[
H^{(E)}_{2,m}(x) = -i \frac{e^{ikr}}{r} \left[ \left( 1 + \frac{6i}{kr} - \frac{15}{(kr)^2} - \frac{15i}{(kr)^3} \right) (n \cdot Q_m \cdot n) \right] \\
- \left( 1 + \frac{3i}{kr} - \frac{6}{(kr)^2} - \frac{6i}{(kr)^3} \right) Q_m \cdot n \right].
\]  

(S.33)
Solutions for Problem 3

Part (a):
The oscillating magnetic field of the incident wave looks approximately uniform from the sphere’s point of view, but it cannot penetrate the sphere itself due to the skin effect. Or rather, it penetrate only to the skin depth, and for a perfectly good conductor skin depth $\to 0$. Thus, the incident magnetic field is screened from the inside of the sphere by the surface currents, just like the incident electric field is screened by the surface charges.

To find the surface currents and hence the net magnetic dipole moment, let’s compare the sphere at hand to a uniformly magnetized spherical permanent magnet and the bound currents on its surface. We saw in my notes on polarization and magnetization (pages 14–15) that inside the spherical magnet the magnetic field is uniform

$$B = \frac{2}{3} \mu_0 M .$$ (S.34)

By the superposition principle, if we add an external uniform field $H_{\text{ext}}$ without changing the magnetization, we would get

$$B_{\text{inside}} = \mu_0 \left( \frac{2}{3} M + H_{\text{ext}} \right) ,$$ (S.35)

so for $M = -\frac{3}{2} H_{\text{ext}}$ the net magnetic field inside the sphere would vanish. For the same magnetization, the net magnetic moment of the sphere would be

$$m = \left( \text{Volume} = \frac{4\pi r^3}{3} \right) M = -2\pi a^3 H_{\text{ext}} .$$ (S.36)

For the conducting sphere at hand, the situation is physically different but mathematically similar: There is no magnetization or bound currents, instead there are conduction currents on the surface of the sphere, but their net effect on the magnetic field inside the sphere is exactly the same — they precisely cancel the external field $H_{\text{ext}} = H_{\text{inc}}$. Consequently, the conduction currents on the sphere’s surface — or rather the amplitudes of these currents — are precisely the same as the bound currents on the surface or a permanent magnet ball which happens to have zero magnetic field inside it in a similar $H_{\text{ext}}$. Therefore, the
magnetic dipole moments are precisely similar in both cases, or rather the amplitude of the conducting sphere’s dipole moment is related to the incident magnetic field’s amplitude by the same formula (S.36) as the magnetic moment of the spherical magnet, thus

\[ m_{\text{conducting sphere}} = -2\pi r^a H_{\text{inc}}. \]  

\( \text{(1)} \)

\textit{Quod erat demonstrandum.}

**Part (b):**

For the incident wave of wavelength \( \lambda \gg a \), we may treat the incident electric field as approximately uniform external electric field \( E_{\text{ext}} \). Also, the response of the conducting sphere to this external field is much faster than \( 1/\omega \), so we may use the electrostatics techniques to find the induced electric dipole moment at any given time. Thus, as explained in any undergraduate textbook — for example, \textit{Introduction to Electrodynamics} by David Griffith, example 3.8, or in \textit{my notes on separation of variables for 352K class} pages 14–15, — there are induced charges on the sphere’s surface

\[ \rho(r, n) = 3\epsilon_0 E_{\text{ext}} \cos \theta \delta(r - a) \]  

(S.37)

and hence net dipole moment

\[ p = \int d^3x \rho(x) x = 4\pi \epsilon_0 a^3 E_{\text{ext}}. \]  

(S.38)

In the context of the incident EM wave, this dipole moment oscillates with amplitude

\[ p_{\text{conducting sphere}} = +4\pi a^3 \epsilon_0 E_{\text{inc}}. \]  

(2)

Next, to relate the electric and the magnetic dipole moments of the conducting sphere to each other, we note that the electric and the magnetic amplitudes of the incident wave
are related to each other as
\[
H_{\text{inc}} = \frac{1}{Z_0} \mathbf{n}_0 \times \mathbf{E}_{\text{inc}} \quad \text{(3.39)}
\]
where \( \mathbf{n}_0 \) is the unit vector in the direction of the incident wave. Consequently,
\[
\frac{\mathbf{m}}{c} = -\frac{2\pi a^3}{c} \mathbf{H}_{\text{inc}} = -\frac{2\pi a^3}{Z_0 c} \mathbf{n}_0 \times \mathbf{E}_{\text{inc}} = -2\pi a^3 \epsilon_0 \mathbf{n}_0 \times \mathbf{E}_{\text{inc}} = -\frac{1}{2} \mathbf{n}_0 \times (4\pi a^3 \epsilon_0 \mathbf{E}_{\text{inc}})
\]
\[
= -\frac{1}{2} \mathbf{n}_0 \times \mathbf{p}.
\quad \text{(3)}
\]

Quod erat demonstrandum.

Part (c):
As I explained in class — see also my notes on magnetic dipole and electric quadrupole radiation — in the radiation zone, far away from the oscillating multipoles the EM fields are
\[
\mathbf{E}_{\text{sc}} = ik Z_0 (\mathbf{n} \times (\mathbf{n} \times \mathbf{f})) \frac{e^{ikr - i\omega t}}{r}, \quad \mathbf{H}_{\text{sc}} = -ik (\mathbf{n} \times \mathbf{f}) \frac{e^{ikr - i\omega t}}{r} \quad \text{(3.40)}
\]
for
\[
\mathbf{f}(\mathbf{n}) = \frac{1}{4\pi} \int d^3 y \mathbf{J}(y) \exp(-i k \mathbf{n} \cdot \mathbf{y}). \quad \text{(3.41)}
\]
Specifically, for an electric dipole of amplitude \( \mathbf{p} \)
\[
\mathbf{f}_{\text{ED}} = \frac{i\omega}{4\pi} \mathbf{p} \quad \text{(3.42)}
\]
while for a magnetic dipole
\[
\mathbf{f}_{\text{MD}}(\mathbf{n}) = -\frac{i\omega}{4\pi c} \mathbf{n} \times \mathbf{m} \quad \text{(3.43)}
\]

For the case at hand, both electric and magnetic oscillating dipoles are present and have comparable magnitudes, or rather \( \mathbf{m}/c \sim \mathbf{p} \). Therefore
\[
\mathbf{f}_{\text{net}} \approx \mathbf{f}_{\text{ED}} + \mathbf{f}_{\text{MD}}(\mathbf{n}) = \frac{i\omega}{4\pi} \left( \mathbf{p} - \mathbf{n} \times \frac{\mathbf{m}}{c} \right). \quad \text{(3.44)}
\]

In light of eqs. (3) and (2), this formula evaluates to
\[
f = \frac{i\omega}{4\pi} \left( \mathbf{p} + \frac{1}{2} \mathbf{n} \times (\mathbf{n}_0 \times \mathbf{p}) \right) = i\omega a^3 \epsilon_0 E_0 \left( \mathbf{e}_0 + \frac{1}{2} \mathbf{n} \times (\mathbf{n}_0 \times \mathbf{e}_0) \right). \quad \text{(3.45)}
\]
Part (d):
As explained in class on 4/9, the polarized partial cross section obtains from \( f(n) \) according to
\[
\frac{d\sigma}{d\Omega} = \frac{k^2 Z_0^2}{E_0^2} \left| e^* \cdot (n \times (n \times f)) \right|^2 = \frac{k^2 Z_0^2}{E_0^2} \left| e^* \cdot f \right|^2.
\] (S.46)

In particular, for \( f(n) \) as in eq. (S.45),
\[
\frac{d\sigma}{d\Omega} = \left( kZ_0 \omega a^3 \epsilon_0 \right)^2 \left| e^* \cdot e_0 + \frac{1}{2} e^* \cdot (n \times (n_0 \times e_0)) \right|^2.
\] (S.47)

To simplify this formula, note that in the first factor
\[
kZ_0 \times \omega a^3 \epsilon_0 = k^2 a^3 \times cZ_0 \epsilon_0 = k^2 a^3 \quad \Rightarrow \quad (kZ_0 \omega a^3 \epsilon_0)^2 = k^4 a^6,
\] (S.48)

while inside \(| \cdots |^2\)
\[
e^* \cdot (n \times (n_0 \times e_0)) = (n_0 \times e_0) \cdot (e^* \times n) = -(n \times e^*) \cdot (n_0 \times e_0).
\] (S.49)

Consequently,
\[
\frac{d\sigma}{d\Omega} = k^4 a^6 \left| e^* \cdot e_0 - \frac{1}{2} (n \times e^*) \cdot (n_0 \times e_0) \right|^2.
\] (S.50)

exactly as in eq. (4).

Next, let’s specialize to linear polarizations \( \perp \) or \( \parallel \) to the scattering plane. Note that if \( e_0 \parallel \) the scattering plane then \((n_0 \times e_0) \perp\) the plane and vice versa; likewise if \( e \parallel \) the plane then \((n \times e) \perp\) the plane and vice versa. Consequently, IF \( e_0 \perp \) the plane while \( e \parallel \) the plane OR IF \( e_0 \parallel \) the plane while \( e \perp \) the plane THEN both \( e^* \cdot e_0 = 0 \) and \( (n \times e^*) \cdot (n_0 \times e_0) = 0 \), and hence \( d\sigma/d\Omega = 0 \). Therefore, if the incident wave is polarized \( \perp \) to the scattering plane, then the scattering wave is also polarized \( \perp \) to the scattering plane, and likewise if the incident wave is polarized \( \parallel \) to the scattering plane, then the scattering wave is also polarized \( \parallel \) to the scattering plane.
Now, suppose both waves are polarized $\perp$ to the plane of scattering. In the coordinate system where $z$ axis points along the incident wave direction $\mathbf{n}_0$ while the $xz$ plane is the scattering plane,

$$\mathbf{n}_0 = (0, 0, 1), \quad \mathbf{n} = (\sin \theta, 0, \cos \theta), \quad (S.51)$$

we have

$$\mathbf{e}_0 = (0, 1, 0), \quad \mathbf{e} = (0, 1, 0), \quad (S.52)$$

hence

$$\mathbf{n}_0 \times \mathbf{e}_0 = (-1, 0, 0), \quad \mathbf{n} \times \mathbf{e} = (-\cos \theta, 0, +\sin \theta),$$

and therefore

$$\mathbf{e}^* \cdot \mathbf{e}_0 = +1, \quad (\mathbf{n} \times \mathbf{e}^*) \cdot (\mathbf{n}_0 \times \mathbf{e}_0) = + \cos \theta. \quad (S.53)$$

Plugging this geometry into eq. (4), we get

$$\frac{d\sigma^\perp}{d\Omega} = k^4 a^6 \left(1 - \frac{1}{2} \cos \theta\right)^2, \quad (S.54)$$

in perfect agreement with the first eq. (5).

Finally, suppose both incident and scattered waves are polarized $\parallel$ to the scattering plane. In this case, in the coordinate system (S.51) we have

$$\mathbf{e}_0 = (1, 0, 0), \quad \mathbf{e} = (\cos \theta, 0, -\sin \theta), \quad (S.55)$$

hence

$$\mathbf{n}_0 \times \mathbf{e}_0 = (0, 1, 0), \quad \mathbf{n} \times \mathbf{e} = (0, 1, 0), \quad (S.56)$$

and therefore

$$\mathbf{e}^* \cdot \mathbf{e}_0 = + \cos \theta, \quad (\mathbf{n} \times \mathbf{e}^*) \cdot (\mathbf{n}_0 \times \mathbf{e}_0) = +1. \quad (S.57)$$

This time, plugging this geometry into eq. (4) yields

$$\frac{d\sigma^\parallel}{d\Omega} = k^4 a^6 \left(\frac{1}{2} - \cos \theta\right)^2, \quad (S.58)$$

in perfect agreement with the second eq. (5).
Quod erat demonstrandum.

Part (e):
Saying that the incident wave is un-polarized means that in any polarization basis half of the net power belongs to one polarization and half to the other. In particular, half of the net incident energy flux belongs to the linear polarization $\parallel$ to the scattering plane and the other half to the polarization $\perp$ to the scattering plane. Consequently, the partial cross-section in which the scattered wave’s polarization is not detected is simply

$$\frac{d\sigma^{\text{unpolarized}}}{d\Omega} = \frac{1}{2} \frac{d\sigma^\parallel}{d\Omega} + \frac{1}{2} \frac{d\sigma^\perp}{d\Omega}. \quad (S.59)$$

Specifically, for the polarized partial cross-sections as in eqs. (5),

$$\frac{d\sigma^{\text{unpolarized}}}{d\Omega} = \frac{k^4a^6}{2} \times \left( \left( \frac{1}{2} - \cos \theta \right)^2 + \left( 1 - \frac{1}{2} \cos \theta \right)^2 \right)$$

$$= \frac{k^4a^6}{2} \times \left( \left( \frac{1}{4} - \cos \theta + \cos^2 \theta \right) + \left( 1 - \cos \theta + \frac{1}{4} \cos^2 \theta \right) \right) \quad (S.60)$$

$$= \frac{k^4a^6}{2} \times \left( \frac{5}{4} - 2 \cos \theta + \frac{5}{4} \cos^2 \theta \right).$$

Note that this partial cross-section does not have a forward-backward symmetry. Instead, the scattering into the backward hemisphere ($\theta > 90^\circ$ so that $\cos \theta < 0$) is significantly stronger than the scattering into the forward hemisphere ($\theta < 90^\circ$ so that $\cos \theta > 0$). Graphically,
Now consider the partial polarization of the scattered wave. Although the incident wave has equal powers of the two polarizations, they scatter with different strengths due to un-equal polarized cross-sections. Consequently, the degree to which the scattered wave is polarized is

$$\Pi(\theta) = \frac{dP_{\perp} - dP_{\parallel}}{dP_{\perp} + dP_{\parallel}} = \frac{d\sigma_{\perp} - d\sigma_{\parallel}}{d\sigma_{\perp} + d\sigma_{\parallel}}.$$ (S.61)

For the polarized cross-sections as in eqs. (5), this formula yields

$$\Pi(\theta) = \frac{(1 - \frac{1}{2} \cos \theta)^2 - (\frac{1}{2} - \cos \theta)^2}{(1 - \frac{1}{2} \cos \theta)^2 + (\frac{1}{2} - \cos \theta)^2} = \frac{\frac{3}{4} - \frac{3}{4} \cos^2 \theta}{\frac{5}{4} - 2 \cos \theta + \frac{5}{4} \cos^2 \theta} \quad \text{(S.62)}$$

Graphically,

Note: at $\theta = 60^\circ$ the scattered wave is 100% polarized $\perp$ to the scattering plane — the $\parallel$ polarization does not scatter in this direction.
Part (e):
Finally, the net scattering cross-section obtains by integrating the unpolarized partial cross-section over the $4\pi$ directions of the scattered wave. Thus

$$
\sigma_{\text{net}} = \int d\Omega \frac{d\sigma^{\text{unpolarized}}}{d\Omega}
$$

$$
= \frac{k^4 a^6}{8} \int d\Omega \left( 5 - 8 \cos \theta + 5 \cos^2 \theta \right)
$$

$$
= \frac{k^4 a^6}{8} \times \left( 5 \times 4\pi - 8 \times 0 + 5 \times \frac{4\pi}{3} \right) = \frac{k^4 a^6}{8} \times \frac{80\pi}{3}
$$

$$
= \frac{10\pi}{3} \times k^4 a^4.
$$