Problem 1(a):
Consider 3 inertial frames, \( K, K', \) and \( K'' \). The \( K' \) frame moves at constant velocity \( \mathbf{v}_1 \) relative to the frame \( K \), while the \( K'' \) frame moves at constant velocity \( \mathbf{v}_2 \) relative to the frame \( K' \). For simplicity, suppose the velocities \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are in the same direction, and let the \( x \) axes in all 3 frames point in that direction.

In this setup, the coordinate transform between the \( K \) and the \( K' \) frames is
\[
x' = \gamma_1 x - \gamma_1 v_1 t, \quad t' = \gamma_1 t - \frac{\gamma_1 v_1}{c^2} x, \quad \gamma_1 = \frac{1}{\sqrt{1 - (v_1/c)^2}},
\]
and there is a similar transform between the \( K' \) and the \( K'' \) frames
\[
x'' = \gamma_2 x' - \gamma_2 v_2 t', \quad t'' = \gamma_2 t' - \frac{\gamma_2 v_2}{c^2} x', \quad \gamma_2 = \frac{1}{\sqrt{1 - (v_2/c)^2}}.
\]
Combining these formulae, we get the direct transform from the \( K \) frame to the \( K'' \) frame,
\[
\begin{align*}
x'' &= \gamma_2 \left( \gamma_1 x - \gamma_1 v_1 t \right) - \gamma_2 v_2 \left( \gamma_1 t - \frac{\gamma_1 v_1}{c^2} x \right) \\
&= \gamma_2 \gamma_1 \left( 1 + \frac{v_2 v_1}{c^2} \right) x - \gamma_2 \gamma_1 (v_2 + v_1) t, \\
t'' &= \gamma_2 \left( \gamma_1 t - \frac{\gamma_1 v_2}{c^2} x \right) - \frac{\gamma_2 v_2}{c^2} (\gamma_1 x - \gamma_1 v_2 t) \\
&= \gamma_2 \gamma_1 \left( 1 + \frac{v_2 v_1}{c^2} \right) t - \frac{\gamma_2 \gamma_1 (v_2 + v_1)}{c^2} x.
\end{align*}
\]
The transform (S.3) can be identified as a Lorentz transform for some relative velocity \( v_{1+2} \), provided we let
\[
\gamma_{1+2} = \gamma_1 \gamma_1 \left( 1 + \frac{v_2 v_1}{c^2} \right) \quad \text{and} \quad \gamma_{1+2} \times v_{1+2} = \gamma_2 \gamma_1 (v_2 + v_1).
\]
Note: the parameters \( v_{1+2} \) and \( \gamma_{1+2} \) of a Lorentz transform are not independent variables, they must be related to each other as
\[
\gamma_{1+2} = \frac{1}{\sqrt{1 - (v_{1+2}/c)^2}}.
\]
Together, eqs. (S.4) and (S.5) make 3 equations for two unknowns, but fortunately there is a unique solution. To find it, we take the ratio of the two eqs. (S.4), which immediately
gives us
\[ v_{1+2} = \frac{v_2 + v_1}{1 + (v_2v_1/c^2)}. \] (1)

Now let’s plug this velocity into eq. (S.5) and make sure the resulting \( \gamma_{1+2} \) is consistent with the first eq. (S.4). Or rather, let’s work out the
\[
\frac{1}{\gamma_{1+2}^2} = 1 - \frac{v_{1+2}^2}{c^2} = 1 - \frac{1}{c^2} \left( \frac{v_2 + v_1}{1 + (v_2v_1/c^2)} \right)^2
\]
\[
= \frac{1 - \frac{c^2(v_1 + v_2)^2}{(c^2 + v_1v_2)^2}}{(c^2 + v_1v_2)^2} = \frac{(c^2 + v_1v_2)^2 - c^2(v_1 + v_2)^2}{(c^2 + v_1v_2)^2}
\]
\[
= \frac{c^4 + 2c^2v_1v_2 + v_1^2v_2^2 - c^2v_1^2 - 2c^2v_1v_2 - c^2v_2^2}{(c^2 + v_1v_2)^2} = \frac{(c^2 - v_1^2) \times (c^2 - v_2^2)}{(c^2 + v_1v_2)^2}
\]
\[
= \frac{1}{\gamma_1^2} \times \frac{1}{\gamma_2^2} \times \frac{c^4}{(c^2 + v_1v_2)^2},
\]
which gives us
\[ \gamma_{1+2} = \gamma_1 \times \gamma_2 \times \left( 1 + \frac{v_1v_2}{c^2} \right). \] (S.7)

Fortunately, this formula is in perfect agreement with the first eq. (S.4), so the relative velocity (1) indeed solves both equations (S.4) and (S.5) for the combined Lorentz transform.

**Problem 1(b):**
Relative to the moving fluid, the light moves at speed
\[ u_0 = \frac{c}{n}. \] (S.8)

The speed of this light relative to the lab obtains from the relativistic velocity addition formula
\[ u = \frac{u_0 \pm v}{1 \pm (vu_0/c^2)} = \frac{c \pm v}{1 \pm \frac{v}{nc}}. \] (S.9)

Expanding this formula in powers of \( v/c \), we obtain
\[ u = \frac{c}{n} \pm \left( 1 - \frac{1}{n^2} \right) \times v + O \left( \frac{v^2}{c} \right), \] (S.10)
where the second-order term \( O(v^2/c) \) is negligibly small. Indeed, in a typical Fizeau-style
experiment, the speed of the moving liquid is a few m/s so that \( v/c \sim 10^{-8} \), which means that the second-order (in \( v/c \)) term would be 8 orders of magnitude smaller than the first order term. Thus, for \( v \ll c \) we get the Fizeau formula
\[
 u = \frac{c}{n} \pm \left( 1 - \frac{1}{n^2} \right) \times v. \tag{2}
\]

**Problem 1(c):**
Suppose the frequency of the light source (which is at rest in the lab frame) is \( \omega_0 \). Due to the Doppler effect, the light going through the receding liquid would have a lower frequency while the light in an in-flowing liquid would have a higher frequency. In the non-relativistic approximation
\[
 \omega' = \omega_0 \times \left( 1 \mp \frac{v}{c/n} \right), \tag{S.11}
\]
while the relativistic corrections due to time dilation, etc., are of the second order in \( v/c \). Neglecting such second-order corrections, we write the frequency of light as seen by the moving fluid
\[
 \omega' \approx \omega_0 \mp \frac{n v \omega_0}{c}. \tag{S.12}
\]

Changing the frequency of the light wave changes the fluid’s refraction index \( n(\omega) \), but since the difference \( \omega' - \omega_0 \) is rather small, we may approximate
\[
 n(\omega') \approx n(\omega_0) + \frac{dn}{d\omega} \times (\omega' - \omega_0) = n(\omega_0) \mp \frac{n v}{c} \times \omega \frac{dn}{d\omega}. \tag{S.13}
\]
Consequently, the speed of light relative to the moving fluid changes from the
\[
 u_0 = \frac{c}{n(\omega_0)} \tag{S.14}
\]
to
\[
 u_0' = \frac{c}{n(\omega')} \approx \frac{c}{n(\omega_0)} - \frac{c}{n^2(\omega_0)} \times (n(\omega') - n(\omega_0)) = \frac{c}{n(\omega_0)} \pm v \times \omega \frac{dn}{n d\omega}. \tag{S.15}
\]

Finally, combining this rather small effect with the “aether drag” (2) — which is also a small effect — we find that to the first order in \( v/c \), the speed of light in the moving liquid
is

\[ u = \frac{c}{n(\omega_0)} \pm v \times \left( 1 - \frac{1}{n^2} + \frac{\omega}{n} \frac{dn}{d\omega} \right). \]  \hspace{1cm} (3)

**Problem 2(a):**
The time on board the spaceship — regardless of whether it’s moving at a constant velocity or accelerating — is the proper time along its worldline. Infinitesimally,

\[(c d\tau)^2 = (c dt)^2 - (d\mathbf{x})^2 = (c^2 - v^2) dt^2, \]  \hspace{1cm} (S.16)

hence

\[ d\tau = \sqrt{1 - (v/c)^2} \times dt. \]  \hspace{1cm} (S.17)

If you are solving this problem before I have explained the proper time in class, here is another explanation: At any particular moment, the time on shipboard runs at the same rate as the time in the inertial frame which happens to be co-moving with the ship at that moment, thus

\[ d\tau = dt' = dt \times \sqrt{1 - (v/c)^2}. \]  \hspace{1cm} (S.18)

Next, consider the ship’s velocity \( v(\tau) \). Consider the inertial frame which at time \( \tau_0 \) has the same velocity \( v_0 = v(\tau_0) \) as the ship. Relative to that frame, \( v'(\tau_0) = 0 \) while at an infinitesimally later time \( \tau_0 + d\tau \)

\[ v'(\tau_0 + d\tau) = ad\tau. \]  \hspace{1cm} (S.19)

By the relativistic velocity addition formula (1), this translates into the velocity relative to
the Earth (or whatever planet the ship was launched from) as

\[ v + dv = \frac{v + ad\tau}{1 - v(ad\tau)/c^2} = v + \left(1 - \frac{v^2}{c^2}\right) ad\tau + O(d\tau^2), \]  
(S.20)

thus

\[ dv(\tau) = \left(1 - \frac{v^2}{c^2}\right) a d\tau \]  
(S.21)

To solve this differential equation, we write it as

\[ \frac{a}{c} d\tau = \frac{d(v/c)}{1 - (v/c)^2} = d \left(\text{ar tanh} \frac{v}{c}\right), \]  
(S.22)

which immediately gives us

\[ \frac{v}{c} = \tanh \left(\frac{a\tau}{c} + \text{const}\right). \]  
(S.23)

In particular, if we set the shipboard clock to zero at the launch time, we get

\[ v(\tau) = c \times \tanh \frac{a\tau}{c}. \]  
(S.24)

The rest of eqs. (7) follow from this formula and from the time equation (S.17). Indeed, consider the relation between Earth time \( t \) and the shipboard time \( \tau \). Infinitesimally,

\[ dt = \frac{d\tau}{\sqrt{1 - (v/c)^2}} = \frac{d\tau}{\sqrt{1 - \tanh^2(a\tau/c)}} = d\tau \times \cosh(a\tau/c), \]  
(S.25)

which easily integrates to

\[ t = \frac{c}{a} \times \sinh \frac{a\tau}{c}. \]  
(S.26)
Finally, to write the velocity $v$ as a function of Earth time $t$, we note that

$$\tanh\left(\frac{a\tau}{c}\right) = \frac{\sinh\left(\frac{a\tau}{c}\right)}{\cosh\left(\frac{a\tau}{c}\right)} = \frac{\sinh\left(\frac{a\tau}{c}\right)}{\sqrt{1 + \sinh^2\left(\frac{a\tau}{c}\right)}} = \frac{(at/c)}{\sqrt{1 + (at/c)^2}} \quad (S.27)$$

where the last equality follows from eq. (S.26). Hence, in light of eq. (S.24),

$$v(t) = \frac{at}{\sqrt{1 + (at/c)^2}}. \quad (S.28)$$

**Problem 2(b):**

Let’s start with the distance traveled by the ship. Infinitesimally,

$$dx = v\, dt = c \, \tanh\left(\frac{a\tau}{c}\right) \times d\left(\frac{c}{a} \sinh\left(\frac{a\tau}{c}\right)\right) = c \, \tanh\left(\frac{a\tau}{c}\right) \times \cosh\left(\frac{a\tau}{c}\right) \times d\tau = c \, \sinh\left(\frac{a\tau}{c}\right) \times d\tau = \frac{c^2}{a} \times d\left(\cosh\left(\frac{a\tau}{c}\right)\right),$$

which integrates to

$$x(\tau) = \frac{c^2}{a} \times \left[\cosh\left(\frac{a\tau}{c}\right) - 1\right]. \quad (S.30)$$

Or in terms of the Earth time,

$$\cosh\left(\frac{a\tau}{c}\right) = \sqrt{1 + \sinh^2\left(\frac{a\tau}{c}\right)} = \sqrt{1 + \left(\frac{at}{c}\right)^2}, \quad (S.31)$$

hence

$$x(t) = \frac{c^2}{a} \times \left[\sqrt{1 + \left(\frac{at}{c}\right)^2} - 1\right]. \quad (S.32)$$
Now consider a light signal sent towards the accelerating ship at time \( t_s > 0 \). At time \( t > t_s \) the signal is at
\[
x_s(t) = c(t - t_s),
\]
so as long as the ship is accelerating at the constant rate \( a \),
\[
\frac{a}{c^2} \times (x(t) - x_s(t)) = \left[ \sqrt{1 + \left( \frac{at}{c} \right)^2} - 1 \right] - \frac{a}{c} (t - t_s)
\]
\[
= \left[ \sqrt{1 + \left( \frac{at}{c} \right)^2} - \frac{at}{c} \right] + \left[ \frac{at}{c} - 1 \right].
\]
On the last line here, the first term is always positive, although it asymptotes to zero for \( t \to \infty \). So unless the second term is negative, the ship would stay ahead of the light signal as long as it keeps accelerating, and the signal would never catch up with the ship!

In other words, to get a light signal — or any other kind of a signal — to the ship while it keeps accelerating, the signal must be send within the time window
\[
0 < t_s < t_{s}^{\text{max}} = \frac{c}{a}.
\]

For a ship accelerating at the rate \( a = g = 9.8 \text{ m/s}^2 \), this time window lasts about 354 days, or approximately 1 year.

**Problem 2(c):**
For a ship accelerating at the rate \( a = g = 9.80 \text{ m/s}^2 \), we have
\[
\frac{c}{g} \approx 354 \text{ days} = 0.969 \text{ year}
\]
and hence
\[
\frac{c^2}{g} = 0.969 \text{ lightyear}.
\]

For the round trip in question, for each of the four segments of the flight — Earth to the midpoint, the midpoint to Gliese 667 Cc, Gliese 667 Cc back to the midpoint, and the
midpoint to Earth, — we have

\[
\frac{gx_{\text{segment}}}{c^2} = \frac{\frac{1}{2}(23.62 \text{ ly})}{0.969 \text{ ly}} \approx 12.2. \quad (\text{S.38})
\]

Comparing this number to eqs. (S.32) and (S.30), we see that each segment takes

\[
t_{\text{segment}} = \frac{c}{g} \times \sqrt{\left(\frac{gx_{\text{segment}}}{c^2} + 1\right)^2 - 1} = (0.969 \text{ yr}) \times \sqrt{(12.2 + 1)^2 - 1} = 12.74 \text{ yr} \quad (\text{S.39})
\]

by the Earthbound clock, and

\[
\tau_{\text{segment}} = \frac{c}{g} \times \text{ar cosh} \left(\frac{gx_{\text{segment}}}{c^2} + 1\right) = (0.969 \text{ yr}) \times \text{ar cosh}(12.2 + 1) = 3.17 \text{ yr} \quad (\text{S.40})
\]

by the shipboard clock.

Counting the 4 segments of the flight and 1 year stop at the Gliese 667 Cc planet, the whole trip lasts

\[
t_{\text{net}} = 4 \times t_{\text{segment}} + 1 \text{ year} = 51.96 \text{ year} \quad (\text{S.41})
\]

by the Earth clock but only

\[
\tau_{\text{net}} = 4 \times \tau_{\text{segment}} + 1 \text{ year} = 13.68 \text{ year} \quad (\text{S.42})
\]

by the shipboard clock. Thus, if the twin who made the crew left Earth on his 21\textsuperscript{th} birthday, he would come back to Earth only about 34\frac{2}{3} years old, quite young. But his brother who stayed on Earth would be almost 72 years old!
Problem 3(a):
For simplicity, assume the particle moves at constant velocity $v'$ relative to the reference frame $K'$, thus $x' = v't'$. Lorentz transforming the space and the time coordinates into the $K$ frame, we get

$$t = \gamma_u t' + \frac{\gamma_u u}{c^2} \cdot x' = \gamma_u t' + \frac{\gamma_u u}{c^2} \cdot v't'$$

$$= \gamma_u \left(1 - \frac{u \cdot v'}{c^2}\right) \times t', \quad (S.43)$$

$$x = x' + \frac{\gamma_u - 1}{u^2} (u \cdot x')u + \gamma_u u t'$$

$$= v't' + \frac{\gamma_u - 1}{u^2} (u \cdot v't')u + \gamma_u u t'$$

$$= \left(v' + \frac{\gamma_u - 1}{u^2} (u \cdot v')u + \gamma_u u\right)t', \quad (S.44)$$

or in terms of the components $v'_\parallel$ and $v'_\perp$ of $v$; that are parallel / perpendicular to $u$,

$$x = \left(v'_\perp + \gamma_u v'_\parallel + \gamma_u u\right)t'. \quad (S.45)$$

Re-expressing this $x$ as a function of $t$ rather that $t'$, we get

$$x(t) = \left(1 + \frac{u \cdot v'}{c^2}\right)^{-1} \left(\frac{v'_\perp}{\gamma_u} + v'_\parallel + u\right)t \quad (S.46)$$

and therefore velocity relative to the $K$ frame

$$v = \left(1 + \frac{u \cdot v'}{c^2}\right)^{-1} \left(\frac{v'_\perp}{\gamma_u} + v'_\parallel + u\right) \quad (5)$$

Or in components parallel and perpendicular to $u$,

$$v'_\parallel = \frac{v'_\parallel + u}{1 + \frac{u \cdot v'}{c^2}}, \quad v'_\perp = \frac{v'_\perp}{\gamma_u \left(1 + \frac{u \cdot v'}{c^2}\right)} \quad (S.47)$$
Problem 3(b):
For a light wave in vacuum, \( \mathbf{v}' = c \mathbf{n}' \), or in components

\[
v'_\parallel = c \cos \theta', \quad v'_\perp = c \sin \theta' \mathbf{n}_\perp.
\] (S.48)

Plugging these components into eqs. (S.47), we get

\[
v_\parallel = \frac{c \cos \theta' + u}{1 + \frac{u}{c} \cos \theta'}, \quad v_\perp = \frac{c \sin \theta' \mathbf{n}_\perp}{\gamma_u \left(1 + \frac{u}{c} \cos \theta'\right)}.
\] (S.49)

Consequently,

\[
v^2 = v^2_\parallel + v^2_\perp = \frac{(c \cos \theta' + u)^2 + c^2 \sin^2 \theta / \gamma_u^2}{\left(1 + \frac{u}{c} \cos \theta'\right)^2},
\] (S.50)

where the numerator expands to

\[
(c \cos \theta' + u)^2 + \frac{c^2 \sin^2 \theta}{\gamma_u^2} = c^2 \cos^2 \theta' + 2cu \cos \theta' + u^2 + (c^2 - u^2) \times \sin^2 \theta' \\
= c^2 + 2cu \cos \theta' + u^2 \times (1 - \sin^2 \theta' = \cos^2 \theta') \\
= (c + u' \cos \theta')^2.
\] (S.51)

At the same time, the denominator in eq. (S.50) is

\[
\left(1 + \frac{u}{c} \cos \theta'\right)^2 = \frac{1}{c^2} (c + u' \cos \theta')^2,
\] (S.52)

hence

\[
v^2 = \frac{\text{numerator}}{\text{denominator}} = c^2.
\] (S.53)

Thus, the speed of light in the \( K \) frame is also \( c \), quod erat demonstrandum.
Problem 3(c):
For an accelerating particle, the relation between its time coordinates in the two inertial frames $K$ and $K'$ is more complicated than eq. (S.43), but for infinitesimal time intervals the relation between $dt$ and $dt'$ is just as in eq. (S.43),

$$dt = \gamma_u \left( 1 + \frac{\mathbf{u} \cdot \mathbf{v}'(t')}{c^2} \right) \times dt'. \quad (S.54)$$

To simplify our notations, let

$$\kappa \overset{\text{def}}{=} 1 + \frac{\mathbf{u} \cdot \mathbf{v}'}{c^2} = 1 + \frac{uv'_\parallel}{c^2}, \quad (S.55)$$

then

$$dt = \gamma_u \kappa(t') dt'. \quad (S.56)$$

Now, in the $K'$ frame of reference $d\mathbf{v}' = a' dt'$, or in components

$$d\mathbf{v}'_\parallel = a'_\parallel dt', \quad d\mathbf{v}'_\perp = a'_\perp dt', \quad (S.57)$$

Translating these infinitesimal changes velocity from the $K'$ frame to the $K$ frame according to eqs. (S.47), we find

$$\frac{dv'_\parallel}{\kappa c^2} = d \left( \frac{u + v'_\parallel}{\kappa} \right) = \frac{dv'_\parallel}{\kappa} - \frac{u + v'_\parallel}{\kappa^2} \times \frac{u}{c^2} dv'_\parallel$$

$$= \frac{dv'_\parallel}{\kappa^2 c^2} \left( \kappa c^2 - (u + v'_\parallel)u = c^2 + u^2 - u^2 - u^2 = c^2 - u^2 \right) \quad (S.58)$$

$$= \frac{dv'_\parallel}{\kappa^2 \gamma_u} = \frac{a'_\parallel dt'}{\kappa^2 \gamma_u} = \frac{a'_\parallel}{\kappa^2 \gamma_u^2} \times \left( \frac{dt'}{\kappa \gamma_u} \right)$$

$$= \frac{a'_\parallel}{\kappa^3 \gamma_u^3} \times dt$$

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and hence

\[ a_\parallel = \frac{a'_\parallel}{\kappa^2\gamma^2 u} = \frac{\left(1 - \frac{u^2}{c^2}\right)^{3/2}}{\left(1 + \frac{u\nu'}{c^2}\right)^3} a'_\parallel. \]  

(6)

As to the normal (to \( u \)) component of the velocity,

\[ dv_\perp = d \left( \frac{v'_\perp}{\gamma\kappa} \right) = \frac{dv'_\perp}{\kappa\gamma} - \frac{v'_\perp}{\kappa^2\gamma} \left( \frac{d\kappa}{c^2} = \frac{u dv'_\parallel}{c^2} \right) \]

\[ = \frac{a'dt'}{\kappa\gamma} - \frac{v'_\perp}{\kappa^2\gamma c^2} u a'_\parallel dt' \]

\[ = \frac{dt'}{\kappa^2\gamma c^2} \left( \kappa c^2 a'_\perp - u a'_\parallel v'_\perp \right). \]  

(S.59)

On the last line here, the expression in (\( \cdots \)) expands to

\[ \kappa c^2 a'_\perp - u a'_\parallel v'_\perp = (c^2 + u v'_\parallel)a'_\perp - u a'_\parallel v'_\perp \]

\[ = c^2 a'_\perp + (u \cdot v')a'_\perp - (u \cdot a')v'_\perp \]

\[ = c^2 a'_\perp + (u \cdot v')a'_\perp - (u \cdot a')v' \]

\[ = c^2 a'_\perp + u \times (a' \times v'), \]  

(S.60)

while the factor before the (\( \cdots \)) on the last line of (S.59) is

\[ \frac{dt'}{\kappa^2\gamma c^2} = \frac{dt}{\kappa^3\gamma^2 c^2}. \]  

(S.61)

Altogether, we get

\[ dv_\perp = \frac{dt}{\kappa^3\gamma^2 c^2} \left( c^2 a'_\perp + u \times (a' \times v') \right) \]  

(S.62)

and hence normal acceleration

\[ a_\perp = \frac{dv_\perp}{dt} = \frac{1}{\kappa^3\gamma^2} \left[ a'_\perp + \frac{u}{c^2} \times (a' \times v') \right] = \frac{\left(1 - \frac{u^2}{c^2}\right)^{3/2}}{\left(1 + \frac{u\nu}{c^2}\right)^3} \left[ a'_\perp + \frac{u}{c^2} \times (a' \times v') \right]. \]  

(7)