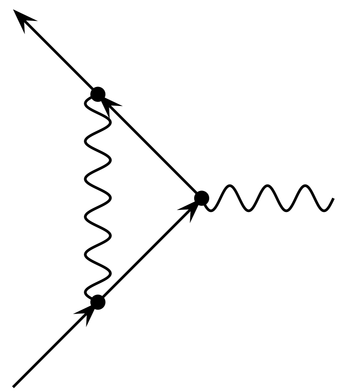
QED Vertex Correction: Working through the Algebra

At the one-loop level of QED, the 1PI vertex correction comes from a single Feynman diagram



(1)

thus

$$ie\Gamma^{\mu}_{1\,\mathrm{loop}}(p',p) \;=\; \int\limits_{\mathrm{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{-ig^{\nu\lambda}}{k^2+i0} \times ie\gamma_\nu \times \frac{i}{\not p' + \not k \; - m + i0} \times ie\gamma^\mu \times \frac{i}{\not p + \not k \; - m + i0} \times ie\gamma_\lambda$$

$$= \; e^3 \int\limits_{\mathrm{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{\mathcal{N}^\mu}{\mathcal{D}} \tag{2}$$

where

$$\mathcal{N}^\mu \;=\; \gamma^\nu(\not k + \not p' + m)\gamma^\mu(\not k + \not p + m)\gamma_\nu \tag{3}$$

and

$$\mathcal{D} \;=\; [k^2 + i0] \times [(p+k)^2 - m^2 + i0] \times [(p'+k)^2 - m^2 + i0]. \tag{4}$$

Using Feynman parameter trick, we re-write the denominator as

$$\frac{1}{\mathcal{D}} \;=\; \iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2}{\Big[x((p+k)^2-m^2) \;+\; y((p'+k)^2-m^2) \;+\; z(k^2) \;+\; i0\Big]^3}\,, \tag{5}$$

and then expand

$$x((p+k)^2-m^2) \;+\; y((p'+k)^2-m^2) \;+\; z(k^2) \;=\; \ell^2 \;-\; \Delta \tag{6}$$

where

$$\ell \;=\; k \;+\; xp \;+\; yp' \tag{7}$$

and

$$\Delta \;=\; (xp+yp')^2 \;+\; x(m^2-p^2) \;+\; y(m^2-p'^2). \tag{8}$$

Using $k^2=(p'-p)^2=p^2+p'^2-2pp'$, we obtain

$$(xp+yp')^2 \;=\; x(x+y)p^2 \;+\; y(x+y)p'^2 \;-\; xyk^2 \tag{9}$$

and hence

$$\Delta \;=\; (1-z)^2m^2 \;-\; xz(p^2-m^2) \;-\; yz(p'^2-m^2) \;-\; xyk^2. \tag{10}$$

For the on-shell electron momenta p and p', this expression simplifies to

$$\Delta \;=\; (1-z)^2m^2 \;-\; xyq^2 \quad \langle\!\langle\text{on shell}\rangle\!\rangle. \tag{11}$$

Altogether, we have

$$\Gamma^{\mu}_{1\,\text{loop}}(p',p) \;=\; -2ie^2 \iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \int\limits_{\text{reg}} \frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}^{\mu}}{\left[\ell^2-\Delta+i0\right]^3}\,, \tag{12}$$

and now we need to simplify the numerator (3) *in the context of this monstrous integral.* The first step is obvious: Let us get rid of the γ^{ν} and γ_{ν} factors using the γ matrix algebra, *eg.,* $\gamma^{\nu}\not{a}\gamma_{\nu}=-2\not{a}$, *etc..* However, in order to allow for the dimensional regularization, we need to re-work the algebra for an arbitrary spacetime dimension D where $\gamma^\nu\gamma_\nu = D \neq 4$. Consequently,

$$\begin{aligned}
\gamma^\nu \not{a} \gamma_\nu &= -2\not{a} + (4-D)\not{a}, \\
\gamma^\nu \not{a}\not{b} \gamma_\nu &= 4(ab) - (4-D)\not{a}\not{b}, \\
\gamma^\nu \not{a}\not{b}\not{c} \gamma_\nu &= -2\not{c}\not{b}\not{a} + (4-D)\not{a}\not{b}\not{c},
\end{aligned} \tag{13}$$

and therefore

$$\mathcal{N}^\mu = -2m^2\gamma^\mu + 4m(p'+p+2k)^\mu - 2(\not{p}+\not{k})\gamma^\mu(\not{p}'+\not{k}) + (4-D)(\not{p}'+\not{k}-m)\gamma^\mu(\not{p}+\not{k}-m). \tag{14}$$

Next, we re-express this numerator in terms of the loop momentum ℓ rather than k using eq. (7). Expanding the result in powers of ℓ, we get quadratic, linear and ℓ–independent terms, but the linear terms do not contribute to the $\int d^D\ell$ integral because they are odd with respect to $\ell \to -\ell$ while everything else in that integral is even. Consequently, *in the context of eq.* (12) we may neglect the linear terms, thus

$$\begin{aligned}
\mathcal{N}^\mu &= -2m^2\gamma^\mu + 4m(p'+p+2\ell-2xp-2yp')^\mu \\
&\quad - 2(\not{p}+\not{\ell}-x\not{p}-y\not{p}')\gamma^\mu(\not{p}'+\not{\ell}-x\not{p}-y\not{p}') \\
&\quad + (4-D)(\not{p}'+\not{\ell}-x\not{p}-y\not{p}'-m)\gamma^\mu(\not{p}+\not{\ell}-x\not{p}-y\not{p}'-m) \\
&\quad \langle\langle\text{skipping terms linear in } \ell\rangle\rangle \\
&\cong -2m^2\gamma^\mu + 4m(p+p'-2xp-2yp')^\mu \\
&\quad - 2\not{\ell}\gamma^\mu\not{\ell} - 2(\not{p}-x\not{p}-y\not{p}')\gamma^\mu(\not{p}'-x\not{p}-y\not{p}') \\
&\quad + (4-D)\not{\ell}\gamma^\mu\not{\ell} + (4-D)(\not{p}'-y\not{p}'-x\not{p}-m)\gamma^\mu(\not{p}-x\not{p}-y\not{p}'-m) \\
&\quad \langle\langle\text{using } p'-p=q \text{ and } x+y+z=1\rangle\rangle \\
&= -2m^2\gamma^\mu + 4mz(p'+p)^\mu + 4m(x-y)q^\mu - (D-2)\not{\ell}\gamma^\mu\not{\ell} \\
&\quad - 2(z\not{p}'+(x-1)\not{q})\gamma^\mu(z\not{p}+(1-y)\not{q}) \\
&\quad + (4-D)(z\not{p}'+x\not{q}-m)\gamma^\mu(z\not{p}-y\not{q}-m).
\end{aligned} \tag{15}$$

Now, let make use of the external fermions being on-shell. This means more than just $p^2 = p'^2 = m^2$: Effectively, we sandwich the vertex $ie\Gamma^\mu$ between Dirac spinors $\bar{u}(p')$ on the left and $u(p)$ on the right. The two spinors satisfy the appropriate Dirac equations $\not{p}u(p) = mu(p)$ and $\bar{u}(p')\not{p}' = \bar{u}(p')m$, so in the context of $\bar{u}(p')\Gamma^\mu u(p)$,

$$A\times\not{p} \;\cong\; A\times m \quad\text{and}\quad \not{p}'\times B \;\cong\; m\times B \tag{16}$$

for any terms in Γ^μ that look like $A\times\not{p}$ or $\not{p}'\times B$ for some A or B. Consequently, the terms on the last two lines of eq. (15) are equivalent to

$$\begin{aligned}(z\not{p}' + (x-1)\not{q})\,\gamma^\mu\,(z\not{p} + (1-y)\not{q}) \;&\cong\; (zm + (x-1)\not{q})\,\gamma^\mu\,(zm + (1-y)\not{q})\\ (z\not{p}' + x\not{q} - m)\,\gamma^\mu\,(z\not{p} - y\not{q} - m) \;&\cong\; ((z-1)m + x\not{q})\,\gamma^\mu\,((z-1)m - y\not{q}).\end{aligned} \tag{17}$$

Let us combine these two expressions with respective coefficients -2 and $4-D$ (*cf.* eq. (15)) and group similar terms together. Making use of

$$\not{q}\gamma^\mu \;=\; q^\mu \;+\; i\sigma^{\mu\nu}q_\nu \quad\text{and}\quad \gamma^\mu\not{q} \;=\; q^\mu \;-\; i\sigma^{\mu\nu}q_\nu\,, \tag{18}$$

we obtain

$$\begin{aligned}&m^2\gamma^\mu\times\Big(-2z^2 + (4-D)(1-z)^2\Big)\\ &+\;\not{q}\gamma^\mu\not{q}\times\Big(2(1-x)(1-y) - (4-D)xy\Big)\\ &+\;mq^\mu\times(x-y)\Big(-2z - (4-D)(1-z)\Big)\\ &+\;im\sigma^{\mu\nu}q_\nu\times\Big(2z(2-x-y) - (4-D)(1-z)(x+y)\Big),\end{aligned} \tag{19}$$

and hence

$$\begin{aligned}\mathcal{N}^\mu \;\cong\; &-(D-2)\,\not{\ell}\gamma^\mu\not{\ell} \;+\; 4mz(p'+p)^\mu\\ &+\; m^2\gamma^\mu\times\Big(-2 - 2z^2 + (4-D)(1-z)^2\Big)\\ &+\;\not{q}\gamma^\mu\not{q}\times\Big(2(1-x)(1-y) - (4-D)xy\Big)\\ &+\; mq^\mu\times(x-y)\Big(4 - 2z - (4-D)(1-z)\Big)\\ &+\; im\sigma^{\mu\nu}q_\nu\times\Big(2z(2-x-y) - (4-D)(1-z)(x+y)\Big).\end{aligned} \tag{20}$$

Furthermore, in the context of the Dirac sandwich $\bar{u}(p')\Gamma^\mu u(p)$ we have

$$\not{q}\gamma^\mu\not{q} \;=\; 2q^\mu\not{q} \;-\; q^2\gamma^\mu \;\cong\; -q^2\gamma^\mu \tag{21}$$

because $\bar{u}(p')\not{q}u(p) = 0$, and also the Gordon identity

$$(p'+p)^\mu \;\cong\; 2m\gamma^\mu \;-\; i\sigma^{\mu\nu}q_\mu\,. \tag{22}$$

Therefore, re-grouping terms and making use of $x+y+z=1$, we obtain

$$\begin{aligned}\mathcal{N}^\mu \;\cong\; &-(D-2)\,\not{\ell}\gamma^\mu\not{\ell} \;+\; m^2\gamma^\mu\times\Big(8z-2(1+z^2)+(4-D)(1-z)^2\Big)\\ &-\; q^2\gamma^\mu\times\Big(2(z+xy)-(4-D)xy\Big) -\; im\sigma^{\mu\nu}q_\nu\times(1-z)\Big(2z+(4-D)(1-z)\Big)\\ &+\; mq^\mu\times(x-y)\Big(4-2z-(4-D)(1-z)\Big).\end{aligned} \tag{23}$$

To further simplify this expression, let us go back to the symmetries of the integral (12). The integral over the Feynman parameters, the integral $\int d^D\ell$, and the denominator $[l^2-\Delta]^3$ are all invariant under the parameter exchange $x\leftrightarrow y$. In eq. (23) for the numerator, the first two lines are invariant under this symmetry, but the last line changes sign. Consequently, only the first two lines contribute to the integral (12) while the third line integrates to zero and may be disregarded.

Finally, thanks to the Lorentz invariance of the $\int d^D\ell$ integral,

$$\ell_\lambda\ell_\nu \;\cong\; g_{\lambda\nu}\times\frac{\ell^2}{D}\,, \tag{24}$$

and hence

$$\not{\ell}\gamma^\mu\not{\ell} \;=\; \gamma^\lambda\gamma^\mu\gamma^\nu\times\ell_\lambda\ell_\nu \;\cong\; \gamma^\lambda\gamma^\mu\gamma^\nu\times g_{\lambda\nu}\frac{\ell^2}{D} \;=\; -(D-2)\gamma^\mu\times\frac{\ell^2}{D}\,. \tag{25}$$

Plugging this formula into eq. (23) and grouping terms according to their γ–matrix structure, we arrive at

$$\mathcal{N}^\mu \;=\; \mathcal{N}_1\times\gamma^\mu \;-\; \mathcal{N}_2\times\frac{i\sigma^{\mu\nu}q_\nu}{2m} \tag{26}$$

where

$$\begin{aligned}\mathcal{N}_1 \;&\cong\; \frac{(D-2)^2}{D}\times\ell^2 \;+\; \Big(8z-2(1+z^2)+(4-D)(1-z)^2\Big)\times m^2\\ &\qquad -\;\Big(2(z+xy)-(4-D)xy\Big)\times q^2\\ &=\; \frac{(D-2)^2}{D}\times\ell^2 \;-\; (D-2)\times\Delta \;+\; 2z\times(2m^2-q^2), \end{aligned}\tag{27}$$

$$\mathcal{N}_2 \;\cong\; (1-z)\Big(4z+2(4-D)(1-z)\Big)\times m^2. \tag{28}$$

Note that splitting the numerator according to eq. (26) is particularly convenient for calculating the electron's form factors:

$$\Gamma^\mu_{1\,\mathrm{loop}} \;=\; F_1^{1\,\mathrm{loop}}(q^2)\times\gamma^\mu \;+\; F_2^{1\,\mathrm{loop}}(q^2)\times\frac{i\sigma^{\mu\nu}q_\nu}{2m}\,, \tag{29}$$

$$F_1^{1\,\mathrm{loop}}(q^2) \;=\; -2ie^2\iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^D\ell}{(2\pi)^D}\,\frac{\mathcal{N}_1}{\big[\ell^2-\Delta+i0\big]^3}\,, \tag{30}$$

$$F_2^{1\,\mathrm{loop}}(q^2) \;=\; +2ie^2\iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^D\ell}{(2\pi)^D}\,\frac{\mathcal{N}_2}{\big[\ell^2-\Delta+i0\big]^3}\,. \tag{31}$$

Electron's Gyromagnetic Moment

As explained earlier in class, electron's spin couples to the static magnetic field as

$$\hat{H}\;\supset\;\frac{-eg}{2m_e}\,\mathbf{S}\cdot\mathbf{B}\qquad\text{where}\quad g\;=\;2\left(F_{\mathrm{mag}}=F_1+F_2\right)\Big|_{q^2=0}\,. \tag{32}$$

The electric form factor $F_1\equiv F_{el}$ for $q^2=1$ is constrained by the Ward identity,

$$F_1^{\mathrm{tot}}\;=\;F_1^{\mathrm{tree}}\,+\,F_1^{\mathrm{loops}}\,+\,F_1^{\mathrm{counter-terms}}\;\xrightarrow[q^2\to 0]{}\;1. \tag{33}$$

Therefore, the gyromagnetic moment is

$$g \;=\; 2 \;+\; 2F_2(q^2=0) \tag{34}$$

where $F_2 = F_2^{\rm loops}$ because the there are no tree-level or counter-term contributions to the F_2, only to the F_1. Thus, to calculate the $g-2$ at the one-loop level, all we need is to evaluate the integral (31) for $q^2=0$.

Let's start with the momentum integral

$$\int\!\frac{d^D\ell}{(2\pi)^D}\,\frac{\mathcal{N}_2}{\bigl[\ell^2-\Delta+i0\bigr]^3} \tag{35}$$

where $\Delta=(1-z)^2m^2$ for $q^2=0$ and $\mathcal{N}_2$ is as in eq. (28). Because the numerator here does not depend on the loop momentum ℓ, this integral converges in $D=4$ dimensions and there is no need for dimensional regularization. All we need is to rotate the momentum into Euclidean space,

$$\begin{aligned}
\int\!\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}_2}{\bigl[\ell^2-\Delta+i0\bigr]^3} \;&=\; \mathcal{N}_2\times\int\!\frac{i\,d^4\ell_E}{(2\pi)^4}\,\frac{1}{-(\ell_E^2+\Delta)^3}\\
&=\;\frac{-i\mathcal{N}_2}{16\pi^2}\times\int\limits_0^\infty\!d\ell_E^2\,\frac{\ell_E^2}{(\ell_E^2+\Delta)^3}\\
&=\;\frac{-i\mathcal{N}_2}{16\pi^2}\times\frac{1}{2\Delta}\\
&=\;\frac{-i}{32\pi^2}\times\frac{\mathcal{N}_2=4z(1-z)m^2\quad\langle\!\langle\text{for } D=4\rangle\!\rangle}{\Delta=(1-z)^2m^2\quad\langle\!\langle\text{for } q^2=0\rangle\!\rangle}\\
&=\;\frac{-i}{32\pi^2}\times\frac{4z}{1-z}\,.
\end{aligned} \tag{36}$$

Substituting this formula into eq. (31), we have

$$F_2^{\rm 1\,loop}(q^2=0) \;=\; \frac{e^2}{16\pi^2}\iiint\limits_0^1\!dx\,dy\,dz\,\delta(x+y+z-1)\times\frac{4z}{1-z}\,. \tag{37}$$

The integrand here depends on z but not on the other two Feynman parameters, so we can immediately integrate over x and y and obtain

$$\iint_0^1 dx\, dy\, \delta(x+y+z-1) \;=\; \int_0^{1-z} dx \;=\; 1-z. \tag{38}$$

Consequently,

$$F_2^{1\,\mathrm{loop}}(q^2=0) \;=\; \frac{e^2}{16\pi^2} \times \int_0^1 dz\,(1-z) \times \frac{4z}{1-z} \;=\; \frac{e^2}{16\pi^2} \times 2 \;=\; \frac{\alpha}{2\pi} \tag{39}$$

and the gyromagnetic moment is

$$g \;=\; 2 \;+\; \frac{\alpha}{\pi} \;+\; O(\alpha^2). \tag{40}$$

Higher-loop calculations are more complicated because the number of diagrams grows very rapidly with the number of loops; at 4-loop order there are thousands of diagrams, and one needs a computer just to count them! Also, at higher orders one has to include effects strong and weak interactions because photons interact not just with electrons and other charged leptons, but also with hadrons and $W^\pm$ particles, which in turn interact with other hadrons, Z^0, Higgs, *etc.*, *etc.* Nevertheless, people have calculated the electron's and muon's g factors up to order α^4 back in the 1970s, and more recent calculations are good up to α^5 order. Meanwhile, the experimentalists have measured g_e to a comparable accuracy of 12 significant digits and g_μ to 9 significant digits

$$g_e \;=\; 2.00231930436\underline{17}(15), \qquad g_\mu \;=\; 2.0023318\underline{16}(12). \tag{41}$$

The theoretical value of g_e is in good agreement with the experimental value, while for the muon there is a small discrepancy $g_\mu^{\mathrm{exp}} - g_\mu^{\mathrm{theory}} \approx (58 \pm 13 \pm 12)\cdot 10^{-10}$. This discrepancy indicates some physics beyond the Standard Model, maybe supersymmetry, maybe something else. In general, effect of heavy particles on g_μ is proportional to $(m_\mu/M_{\mathrm{heavy}})^2$, that's why g_μ is much more sensitive to new physics than g_e.

I would like to complete these notes by calculating $F_2^{1\,\text{loop}}(q^2)$ for $q^2 \neq 0$. Proceeding as in eq. (36) but letting $\Delta = (1-z)^2 m^2 - xyq^2$, we have

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\mathcal{N}_2}{\left[\ell^2 - \Delta + i0\right]^3} \; = \; \frac{-i}{32\pi^2} \times \frac{4z(1-z)m^2}{(1-z)^2 m^2 - xyq^2} \tag{42}$$

and hence

$$F_2^{1\,\text{loop}}(q^2) \; = \; \frac{e^2}{16\pi^2} \iiint_0^1 dx\, dy\, dz\, \delta(x+y+z-1) \times \frac{4z(1-z)m^2}{(1-z)^2 m^2 - xyq^2}\,. \tag{43}$$

To evaluate this integral over Feynman parameters, we change variables from x, y, z to $w = 1 - z$ and $\xi = x/(x+y)$,

$$x \; = \; w\xi, \quad y \; = \; w(1-\xi), \quad z \; = \; 1 - w, \qquad dx\, dy\, dz\, \delta(x+y+z-1) \; = \; w\, dw\, d\xi. \tag{44}$$

Consequently,

$$\begin{aligned} F_2^{1\,\text{loop}}(q^2) \; &= \; \frac{e^2}{16\pi^2} \int_0^1 d\xi \int_0^1 dw\, w \times \frac{4(1-w)w \times m^2}{w^2 \times m^2 \; - \; w^2 \xi(1-\xi) \times q^2} \\ &= \; \frac{e^2}{16\pi^2} \int_0^1 d\xi \frac{m^2}{m^2 \; - \; \xi(1-\xi) q^2} \times \int_0^1 dw\, w \times \frac{4w(1-w)}{w^2} \\ &= \; \frac{e^2}{8\pi^2} \times \int_0^1 d\xi \frac{m^2}{m^2 \; - \; \xi(1-\xi) q^2} \\ &= \; \frac{\alpha}{2\pi} \times \frac{4m^2}{\sqrt{q^2 \times (4m^2 - q^2)}} \times \arctan \sqrt{\frac{q^2}{4m^2 - q^2}}\,. \end{aligned} \tag{45}$$

For $q^2 < 0$ and $-q^2 \gg m^2$,

$$F_2^{1\,\text{loop}}(q^2) \; \approx \; \frac{\alpha}{2\pi} \times \frac{2m^2}{-q^2} \times \log \frac{-q^2}{m^2}\,. \tag{46}$$