Feynman Propagator of a Scalar Field

Earlier in class, I have defined the Feynman propagator of a free real scalar field as a time-ordered correlation function of two scalar fields in the vacuum state,

$$G_F(x-y) \overset{\text{def}}{=} \langle 0 | T \hat{\Phi}(x) \hat{\Phi}(y) | 0 \rangle. \quad (1)$$

We saw that

$$G_F(x-y) = \theta(x^0 > y^0) \times D(x-y) + \theta(x^0 < y^0) \times D(y-x) = \begin{cases} D(x-y) & \text{when } x^0 > y^0, \\ D(y-x) & \text{when } x^0 < y^0, \end{cases} \quad (2)$$

where

$$D(x-y) \overset{\text{def}}{=} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \times \exp \left( -ik(x-y) \right) \delta^{+}\omega_k. \quad (3)$$

A complex scalar field has a similar propagator, but the correlation function involves one $\Phi$ field and one $\Phi^\dagger$ field,

$$\langle 0 | T \hat{\Phi}^\dagger(x) \hat{\Phi}(y) | 0 \rangle = \langle 0 | T \hat{\Phi}(x) \hat{\Phi}^\dagger(y) | 0 \rangle = G_F(x-y). \quad \langle \text{same } G_F \text{ as for a real scalar} \rangle \quad (4)$$

In these notes, I shall show that the propagator (1) is a Green’s function of the Klein–Gordon equation, and then I shall explain why there are many different Green’s functions and which particular Green’s function happens to be the Feynman propagator.

**The Feynman Propagator is a Green’s Function**

A free scalar field obeys the Klein–Gordon equation $(\partial^2 + m^2)\hat{\Phi}(x) = 0$. Consequently, the Feynman propagator (1) for the $\hat{\Phi}$ is a Green’s function of that equation,

$$(\partial^2 + m^2)G_F(x-y) = -i\delta^{(4)}(x-y). \quad (5)$$

Note the delta-function on the RHS is in all four dimensions of the spacetime.
To prove eq. (5), we start with a Lemma: the time derivative of a time-ordered product of two operators \( \hat{A}(t) \) and \( \hat{B}(t_0) \) obtains as

\[
\frac{\partial}{\partial t}(T\hat{A}(t)\hat{B}(t_0)) = T\left(\frac{\partial \hat{A}(t)}{\partial t}\right)\hat{B}(t_0) + \delta(t-t_0) \times [\hat{A}(t), \hat{B}(t_0)].
\]  

(6)

Proof (of the lemma):

\[
T\hat{A}(t)\hat{B}(t_0) \overset{\text{def}}{=} \theta(t > t_0) \times \hat{A}(t)\hat{B}(t_0) + \theta(t < t_0) \times \hat{B}(t_0)\hat{A}(t), \quad (7)
\]

\[
\frac{\partial}{\partial t} \theta(t > t_0) = +\delta(t-t_0), \quad \frac{\partial}{\partial t} \theta(t < t_0) = -\delta(t-t_0), \quad (8)
\]

therefore

\[
\frac{\partial}{\partial t} \left( T\hat{A}(t)\hat{B}(t_0) \right) = \frac{\partial}{\partial t} \left( \theta(t > t_0) \times \hat{A}(t)\hat{B}(t_0) \right) + \frac{\partial}{\partial t} \left( \theta(t < t_0) \times \hat{B}(t_0)\hat{A}(t) \right)
\]

\[
= \delta(t-t_0) \times \hat{A}(t) \times \hat{B}(t_0) + \theta(t > t_0) \times \frac{\partial \hat{A}(t)}{\partial t} \times \hat{B}(t_0)
\]

\[
- \delta(t-t_0) \times \hat{B}(t_0) \times \hat{A}(t) + \theta(t < t_0) \times \hat{B}(t) \times \frac{\partial \hat{A}(t)}{\partial t}
\]

\[
\langle \text{reorganizing terms} \rangle
\]

\[
= \delta(t-t_0) \times \left( \hat{A}(t)\hat{B}(t_0) - \hat{B}(t_0)\hat{A}(t) \right)
\]

\[
+ \left( \theta(t > t_0) \frac{\partial \hat{A}(t)}{\partial t} \hat{B}(t_0) + \theta(t < t_0)\hat{B}(t_0) \frac{\partial \hat{A}(t)}{\partial t} \right)
\]

\[
= \delta(t-t_0) \times [\hat{A}(t), \hat{B}(t_0)] + T\left( \frac{\partial \hat{A}(t)}{\partial t} \hat{B}(t_0) \right).
\]  

(9)

Q.E.D.

Now let’s prove that the propagator (1) is a Green’s function. In light of the lemma (6),

\[
\frac{\partial}{\partial x^0} G_F(x-y) = \langle 0 | \frac{\partial}{\partial x^0} (T\hat{\Phi}(x) \hat{\Phi}(y)) | 0 \rangle
\]

\[
= \langle 0 | T (\partial_0 \hat{\Phi}(x) \times \hat{\Phi}(y)) | 0 \rangle + \delta(x^0-y^0) \times \langle 0 | [\hat{\Phi}(x), \hat{\Phi}(y)] | 0 \rangle.
\]  

(10)

In the second term on the bottom line here, the quantum fields \( \hat{\Phi}(x) \) and \( \hat{\Phi}(y) \) are at equal times \( x^0 = y^0 \), so they commute with each other. Consequently, the second term vanishes,
and we are left with
\[ \frac{\partial}{\partial x^0} G_F(x - y) = \langle 0 | \mathbf{T} \left( \partial_0 \hat{\Phi}(x) \times \hat{\Phi}(y) \right) | 0 \rangle. \] (11)

Now let’s take another time derivative. Again, using the lemma (6), we obtain
\[ \partial_{x^0}^2 G_F(x - y) = \frac{\partial}{\partial x^0} \left( \langle 0 | \mathbf{T} \left( \partial_0 \hat{\Phi}(x) \times \hat{\Phi}(y) \right) | 0 \rangle \right) = \langle 0 | \mathbf{T} \left( \partial_{x^0}^2 \hat{\Phi}(x) \times \hat{\Phi}(y) \right) | 0 \rangle + \delta(x^0 - y^0) \times \langle 0 | \left[ \partial_0 \hat{\Phi}(x), \hat{\Phi}(y) \right] | 0 \rangle. \] (12)

This time, in the second term on the bottom line, \( \partial_0 \hat{\Phi}(x) = \hat{\Pi}(x) \), and at equal times \( x^0 = y^0 \) it does not commute with the \( \hat{\Phi}(y) \). Instead,
\[ \text{for } x^0 = y^0, \quad \left[ \hat{\Pi}(x), \hat{\Phi}(y) \right] = -i\delta^{(3)}(x - y), \] (13)
hence
\[ \delta(x^0 - y^0) \times \langle 0 | \left[ \partial_0 \hat{\Phi}(x), \hat{\Phi}(y) \right] | 0 \rangle = -i\delta^{(3)}(x - y) \times \delta(x^0 - y^0) = -i\delta^{(4)}(x - y). \] (14)

Thus, eq. (12) reduces to
\[ \partial_{x^0}^2 G_F(x - y) = \langle 0 | \mathbf{T} \left( \partial_{x^0}^2 \hat{\Phi}(x) \times \hat{\Phi}(y) \right) | 0 \rangle - i\delta^{(4)}(x - y). \] (15)

Now consider the space-derivative terms in the Klein-Gordon equation. Since the space derivatives commute with the time-ordering,
\[ \nabla_x^2 G_F(x - y) = \nabla_x^2 \langle 0 | (\mathbf{T} \hat{\Phi}(x) \times \hat{\Phi}(y)) | 0 \rangle = \langle 0 | \mathbf{T} \left( \nabla^2 \hat{\Phi}(x) \times \hat{\Phi}(y) \right) | 0 \rangle \] (16)
without any extra terms. Combining this formula with eq. (15), we obtain
\[ (\partial_{x^0}^2 - \nabla^2 + m^2) G_F(x - y) = \langle 0 | \mathbf{T} \left( (\partial_{x^0}^2 - \nabla^2 + m^2) \hat{\Phi}(x) \times \hat{\Phi}(y) \right) | 0 \rangle - i\delta^{(4)}(x - y). \] (17)

On the RHS of this formula, the quantum field \( \hat{\Phi}(x) \) obeys the Klein–Gordon equation \((\partial_{x^0}^2 - \nabla^2 + m^2) \hat{\Phi}(x) = 0\), which kills the first term. Only the second term — the delta function — survives on the RHS, thus
\[ (\partial_{x^0}^2 - \nabla^2 + m^2) G_F(x - y) = -i\delta^{(4)}(x - y), \] (18)
which proves that \( G_F(x - y) \) is a Green’s function of the Klein–Gordon equation. \textbf{Q.E.D.}
**General Green’s functions and the Feynman’s choice**

In general, the same differential equation may have many different Green’s functions, depending on the boundary conditions, etc. So let’s consider a generic Green’s function of the Klein–Gordon equation, that is, some function $G(x - y)$ satisfying

$$(\partial^2 + m^2)G(x - y) = -i\delta^{(4)}(x - y).$$

Let’s Fourier transform this function in all four dimensions, thus

$$G(x - y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \times \tilde{G}(k).$$

In the 4-momentum space, eq. (19) becomes

$$(-k^2 + m^2) \times \tilde{G}(k) = -i,$$

hence naively

$$G(k) = \frac{i}{k^2 - m^2}$$

and therefore

$$G(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2 - m^2}.$$  

The problem with this naive formula is that it integrates over the singularities of the integrand. Indeed, the denominator $k^2 - m^2 = k^2_0 - k^2 - m^0$ vanishes on the mass shells $k^0 = \pm \sqrt{k^2 + m^2}$, so we have two 3D families of poles. In general, an integral of a singular function over its pole is ill-defined, and we must regularize it to get a definite answer. For the Green’s function in question, we must regulate two 3D-families of poles, thus

$$G(x) = \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2 - m^2} = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \times \int \frac{dk_0}{2\pi} \frac{ie^{-ikt_0}}{k_0^2 - k^2 - m^2}. $$

In other words, we integrate over the $k^0$ before we integrate over the $k$. In the $\int dk^0$ integral, we encounter two simple poles at $k^0 = \pm \omega_k$, and we must somehow regularize them to get a definite result. Only then we integrate that result over $k$; hopefully, that integral does not encounter any singularities.
Alas, the devil is in the details: There are many different ways to regularize an integral, and regulators yield different regularized integrals — which eventually yield many different Green’s functions (24) of the same Klein–Gordon equation.

In these notes, we are going to use a particularly simple way to regulate an integral over a simple pole — move the pole away from the real axis into the complex plane,

\[
\int dx \frac{f(x)}{x - x_0} = \int dx \frac{f(x)}{x - (x_0 \pm i\epsilon)}
\]

for an infinitesimal \(\epsilon \to +0\). Equivalently, we may leave the pole real but deform the integration contour slightly away from the real axis so that it bypasses the pole,

\[
\int dx f(x)\]

or

\[
\int dx f(x)
\]

Note that the contour above the pole and the contour below the pole — or equivalently, shifting the pole below or above the real axis — makes for a different regulator which produces a different regularized integral:

\[
\int dx \frac{f(x)}{x - (x_0 + i\epsilon)} - \int dx \frac{f(x)}{x - (x_0 - i\epsilon)} = 2\pi i \times f(x_0).
\]

In the context of the integral (24), there are two poles in the \(\int dk^0\) for every \(k\), so we must make our choices. For the sake of Lorentz invariance, we should use the same regulator for every \(k\), which leaves with \(2 \times 2 = 4\) choices:

- Move the pole at \(k^0 = +\omega_k\) to \(+\omega_k + i\epsilon\) or to \(+\omega_k - i\epsilon\).
- Move the pole at \(k^0 = -\omega_k\) to \(-\omega_k + i\epsilon\) or to \(-\omega_k - i\epsilon\).
The 4 choices give rise to 4 distinct Lorentz-invariant Green’s functions, namely:

1. *Causal retarded Green’s function* $G_R$ for poles at $k_0 \pm \omega_k - i\epsilon$,

\[
\cdots \quad \bullet \quad \bullet \quad \cdots
\]

2. *Causal advanced Green’s function* $G_A$ for poles at $k_0 \pm \omega_k + i\epsilon$,

\[
\cdots \quad \bullet \quad \bullet \quad \cdots
\]

3. *Time-ordered Green’s function* $G_F$ for poles at $k_0 \pm (\omega_k - i\epsilon)$,

\[
\cdots \quad \bullet \quad \cdots
\]

This Green’s function is the Feynman’s propagator (1).

4. *Anti-time-ordered Green’s function* $G_{AT}$ for poles at $k_0 = \pm (\omega_k + i\epsilon)$,

\[
\cdots \quad \bullet \quad \cdots
\]

**Feynman’s Choice**

Let’s focus on the Feynman’s choice of the poles at $+\omega_k - i\epsilon$ and $-\omega_k + i\epsilon$. Altogether, the denominator of the integrand in eq. (24) is

\[
(k_0 - \omega_k + i\epsilon) \times (k_0 + \omega_k - i\epsilon) = k_0^2 - (\omega_k - i\epsilon)^2 \approx k_0^2 - \omega_k^2 + 2i\omega_k \epsilon = k_0^2 - k^2 - m^2 + i\epsilon \times 2\omega_k.
\]

(27)

In the last expression, we may replace $\epsilon \times 2\omega_k$ with simply $\epsilon$, since all we care about is that
it’s a positive infinitesimal number $\to +0$. Thus

\[
\text{denominator } = k_0^2 - k^2 - m^2 + i\epsilon = k^2 - m^2 + i\epsilon, \tag{28}
\]

hence manifestly Lorentz invariant expression for the Feynman’s Green’s function as

\[
G_F(x - y) = \frac{d^4k}{(2\pi)^4} \frac{ie^{-ikx}}{k^2 - m^2 + i\epsilon}. \tag{29}
\]

In this section of the notes, we shall see that this Green’s function is precisely the Feynman propagator (1). Without loss of generality, let’s set $y = 0$. In light of eq. (2), we expect two different cases according to the sign of the $t = x^0$. Let’s start with the $t > 0$ case and deal with $t < 0$ later.

We begin to evaluate the 4D integral (29) by integrating over the $k_0$ for a fixed $k$,

\[
I(t, \omega_k) = \int \frac{dk_0}{2\pi} \frac{ie^{-itk_0}}{k_0^2 - \omega_k^2 + i\epsilon}, \tag{30}
\]

then

\[
G_F(x, t) = \frac{d^3k}{(2\pi)^3} e^{ix \cdot k} \times I(t, \omega_k) \tag{31}
\]

In the integral (30), the integration contour is the real axis, while the two poles lie near the axis — but not quite on it — as on the following diagram

Outside the real axis, the exponential $e^{-itk_0}$ — with positive $t$ — rapidly decreases for large negative $\text{Im}(k_0)$. Consequently, we may close the integration contour by adding to it a large semicircular arc in the negative $\text{Im}(k_0)$ half of the complex plane. Thus,

\[
I(t, \omega_k) = \oint \frac{dk_0}{2\pi} \frac{ie^{-itk_0}}{(k_0 - \omega_k + i\epsilon)(k_0 + \omega_k - i\epsilon)} \tag{33}
\]
The closed-contour integrals like (33) may be evaluated in terms of residues at the poles surrounded by the contour. For the contour (34) at hand, the pole at $+\omega_k - i\epsilon$ lies inside the contour while the other pole lies outside the contour. Consequently,

$$I(t, \omega_k) = -2\pi i \times \text{Residue at } k_0 = +\omega_k - i\epsilon,$$

where the overall $-2\pi i$ factor is due to clockwise direction of the contour. Specifically,

$$I(t, \omega_k) = -2\pi i \times \left( \frac{ie^{-ikt_0}}{2\pi \times (k_0 - \omega_k + i\epsilon) \times (k_0 + \omega_k - i\epsilon)} \right)_{k_0 = +\omega_k - i\epsilon}$$

$$= \frac{\exp(-it(\omega_k - i\epsilon))}{2(\omega_k - i\epsilon)} \langle \langle \text{taking the } \epsilon \to +0 \text{ limit, which is non-singular} \rangle \rangle$$

$$= + \frac{e^{-it\omega_k}}{2\omega_k}.$$ 

Plugging this result into eq. (31), we have

$$G_F(x) = \int \frac{d^3k}{(2\pi)^3} e^{ix \cdot k} \times \frac{e^{-it\omega_k}}{2\omega_k} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \exp(i x \cdot k - it\omega_k) = D(x),$$

in perfect agreement with the Feynman propagator (1) for $t > 0$, cf. eq. (2).
Now let’s turn to the $t < 0$ case. Again, we need to take the integral

$$I(t, \omega_k) = \int \frac{dk_0}{2\pi} \frac{ie^{-ikt_0}}{k_0^2 - \omega_k^2 + i\epsilon}$$

(30)

along the real axis, bypassing the poles according to

$$\cdots \quad \text{\huge .} \quad \cdots$$

(32)

However, for a negative $t$, the exponential $e^{-ikt_0}$ decreases for large positive $\text{Im}(k_0)$ (rather than large negative $\text{Im}(k_0)$ as we had for positive $t$), so to close the integration contour (32) we should add a large semicircular arc in the positive half of the complex plane. Thus,

$$I(t, \omega_k) = \oint_{\Gamma'} \frac{dk_0}{2\pi} \frac{ie^{-ikt_0}}{(k_0 - \omega_k + i\epsilon)(k_0 + \omega_k - i\epsilon)}$$

(38)

where

$$\Gamma'$$

(39)

Unlike the contour (34) which we have used for positive $t$, the contour (39) surrounds the
negative-frequency pole at $k_0 = -\omega_k + i\epsilon$. It is also counterclockwise, hence

$$I(t, \omega_k) = +2\pi i \times \text{Residue at } k_0 = -\omega_k + i\epsilon$$

$$= +2\pi i \times \left( \frac{ie^{-ikt_0}}{2\pi \times (k_0 - \omega_k + i\epsilon) \times (k_0 + \omega_k - i\epsilon)} \right)_{k_0 = -\omega_k + i\epsilon}$$

$$= -\frac{\exp(-it(-\omega_k + i\epsilon))}{2(-\omega_k + i\epsilon)} \langle \langle \text{taking the } \epsilon \to +0 \text{ limit, which is non-singular} \rangle \rangle$$

$$= +\frac{e^{+it\omega_k}}{2\omega_k}.$$ (40)

Plugging this $k_0$ integral into the $\int d^3k$ integral (31), we obtain

$$G_F(x, t) = \int \frac{d^3k}{(2\pi)^3} e^{+ix \cdot k} \times \frac{e^{+it\omega_k}}{2\omega_k} = D(+x, -t).$$

At first blush, this is not quite the answer we want, but fortunately $D$ is invariant under orthochronous Lorentz transformation, and in particular under any rotations of the 3D space. Consequently

$$D(+x, -t) = D(-x, -t),$$ (41)

and therefore

$$\text{for } t < 0, \quad G_F(x) = D(-x),$$ (42)

in perfect agreement with eq. (2).

Altogether, eqs. (37) and (42) tell us that the Feynman’s Green’s function

$$G_F(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ikx}}{k^2 - m^2 + i\epsilon} = \begin{cases} D(x - y) \text{ when } x^0 > y^0 \\ D(y - x) \text{ when } x^0 < y^0 \end{cases} = \langle 0 | \hat{T}\Phi(x)\hat{\Phi}(y) | 0 \rangle$$

is precisely the time-ordered correlation function of two free scalar fields.
Other Green’s functions

Besides the Feynman’s time-ordered Green’s function, there are other useful Green’s functions (of the same Klein-Gordon equation) which obtain for other choices of regularizing the poles. Of particular interest is the causal retarded Green’s function

\[
G_R(x - y) = \int \frac{d^3x}{(2\pi)^3} e^{i(x-y)\mathbf{k}} \times \int \frac{dk_0}{2\pi} \frac{i e^{-i(x^0-y^0)k_0}}{(k_0 - \omega_\mathbf{k} + i\epsilon)(k_0 + \omega_\mathbf{k} + i\epsilon)}, \tag{44}
\]

which obtains by shifting both poles below the real axis,

\[
(45)
\]

As before, we close this contour by adding a large semicircular arc in the lower or upper half of the complex plane, depending on the sign of the time difference \( t = x^0 - y^0 \). In particular, for \( t < 0 \) we close the contour above the real axis,

\[
(46)
\]

which puts both poles outside the contour. Consequently, the contour integral vanishes altogether, thus

\[
G_R(x - y) = 0 \quad \text{when } x^0 - y^0 < 0. \tag{47}
\]

This is why this Green’s function is called \textit{retarded}: time-wise, the point \( x \) must follow the
point \( y \), hence in the context of a source \( j(y) \) and the induced field

\[
\phi(x) = \int d^4 y \, G_R(x - y) \times j(y),
\]

(48)

the source at point \( y \) affects the field \( \phi(x) \) only at later times \( x^0 > y^0 \) that the field.

Now let’s see what \( G_R(x - y) \) looks like for \( t = x^0 - y^0 > 0 \). This time, we close the contour (45) below the real axis,

\[
\Gamma = \quad (49)
\]

so both poles are inside the contour. Consequently,

\[
I_R(t, \omega) = \oint_{\Gamma} \frac{dk_0}{2\pi} \left( \frac{i e^{-itk_0}}{(k_0 - \omega + i\epsilon)(k_0 + \omega + i\epsilon)} \right)
\]

\[
= -2\pi i \times \text{Residue @}(k_0 = +\omega - i\epsilon) - 2\pi i \times \text{Residue @}(k_0 = -\omega - i\epsilon)
\]

\[
= \frac{-2\pi i}{2\pi} \times \left( \frac{ie^{-itk_0}}{(k_0 - \omega + i\epsilon)(k_0 + \omega + i\epsilon)} \right)_{k_0=+\omega-i\epsilon}
\]

\[
+ \frac{-2\pi i}{2\pi} \times \left( \frac{ie^{-itk_0}}{(k_0 - \omega + i\epsilon)(k_0 + \omega + i\epsilon)} \right)_{k_0=-\omega-i\epsilon}
\]

\[
= \frac{e^{-it(\omega-i\epsilon)}}{2(\omega-i\epsilon)} + \frac{e^{-it(-\omega-i\epsilon)}}{2(-\omega-i\epsilon)}
\]

\[
= \frac{e^{-it\omega}}{2\omega} - \frac{e^{+it\omega}}{2\omega}.
\]

(50)
Plugging this integral over the $d^3k$, we obtain

For $x^0 > y^0$, \( G_R(x - y) = \int \frac{d^3k}{(2\pi)^3} e^{i k (x - y)} \times \frac{e^{-i t \omega_k} - e^{+i t \omega_k}}{2\omega_k} = D(x - y; t) - D(x - y; -t) \)

\[ = D(x - y; t) - D(y - x; -t) \]

\[ = D(x - y) - D(y - x). \]  \hspace{1cm} (51)

Note that the bottom line here vanishes for spacelike \((x - y)\), which makes the Green’s function \( G_R \) not only retarded but also causal: it vanishes unless \( y \) lies in the future light cone from \( x \).

Similar to the causal retarded Green’s function \( G_R(x - y) \) we can make the causal advanced Green’s function \( G_A(x - y) \) by shifting both poles above the real axis,

\[ G_A(x - y) = \int \frac{d^3x}{(2\pi)^3} e^{i (x - y) \cdot k} \times \int \frac{dk_0}{2\pi} \frac{i e^{-i(x^0 - y^0)k_0}}{(k_0 - \omega_k - i\epsilon)(k_0 + \omega_k - i\epsilon)} \] \hspace{1cm} (52)

As its name suggests, this Green’s function vanishes unless \( y \) is in the past light cone from \( x \).

Finally, the fourth choice of regularized poles

produce the anti-time-ordered Green’s function

\[ G_{AT}(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2 - m^2 - i\epsilon} = \begin{cases} -D(y - x) & \text{when } x^0 > y^0, \\ -D(x - y) & \text{when } y^0 > x^0. \end{cases} \] \hspace{1cm} (55)
Propagators for Non-scalar Fields

Let me conclude these notes with a few words about propagators for the non-scalar relativistic fields — the vector fields, the tensor fields, the spinor fields, etc., etc. For all such fields, the Feynman propagator is the time-ordered correlation function of two free fields in the vacuum state, for example

\[ G_{F}^{\mu\nu}(x - y) = \langle 0 | T^* A^\mu(x) \times A^\nu(y) | 0 \rangle \] (56)

for the massive vector fields (see homework set #5 for details), or

\[ S_{F}^{\alpha\beta}(x - y) = \langle 0 | T \bar{\Psi}^\alpha(x) \times \Psi^\beta(y) | 0 \rangle \] (57)

for the Dirac spinor field \( \Psi^\beta(x) \) and its conjugate \( \bar{\Psi}^\alpha(x) \) (to be explained in future classes).

All such propagators are Green’s functions of the equations of motion for the appropriate fields. For example, the free massive vector fields obey

\[ \left( g_{\mu\nu}(\partial^2 + m^2) - \partial_\mu \partial_\nu \right) A^\nu = 0, \] (58)

so the propagator is a Green’s function of the differential operator here,

\[ \left( g_{\mu\nu}(\partial^2 + m^2) - \partial_\mu \partial_\nu \right) G_{F}^{\nu\lambda} = -i \delta^\lambda_\mu \times \delta^{(4)}(x - y). \] (59)

(The proof is part of homework set #5.) Likewise, the free Dirac spinor fields \( \Psi^\alpha(x) \) obey the Dirac equation

\[ (i \gamma^\mu \partial_\mu - m)_{\alpha\beta} \Psi^\beta(x) = 0, \] (60)

so the Dirac propagator is a Green’s function of the Dirac equation,

\[ (i \gamma^\mu \partial_\mu - m)_{\alpha\beta} S_{F}^{\beta\delta}(x - y) = -i \delta^\delta_\alpha \times \delta^{(4)}(x - y). \] (61)

(I shall prove this in class in a few weeks.)
Moreover, all such Green’s functions involve momentum integrals over poles along both mass shells $k_0 = \pm \omega_k$, and those poles must be regularized. But for all the Feynman propagators, the poles must be regularized just as we did for the scalar field, the pole at $k_0 = + \omega_k$ shifts below the real axis to $+ \omega_k - i \epsilon$ while the pole at $k_0 = - \omega_k$ shifts above the real axis to $- \omega_k + i \epsilon$. Consequently, all the Feynman propagators have momentum-space form of

$$ (\text{propagator})^{\text{indices}}(x - y) = \int \frac{d^4 k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} \times F^{\text{indices}}(k) \quad (62) $$

for some simple — and hopefully non-singular — function $F^{\text{indices}}(k)$. For example, for the massive vector field

$$ G_{\mu\nu}^F(x - y) = \int \frac{d^4 k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} \times \left(-g^{\mu\nu} - \frac{k^{\mu} k^{\nu}}{m^2}\right), \quad (63) $$

while for the Dirac spinor field

$$ S^{\alpha\beta}_F(x - y) = \int \frac{d^4 k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} \times (k^{\mu} \gamma_{\mu} + m)^{\alpha\beta}. \quad (64) $$

In general, for a massive field the function $F^{\text{indices}}(k)$ is simply a polynomial of $k$ of degree $2 \times \text{Spin}$. For a massless spin $= \frac{1}{2}$ field $F^{\text{indices}}(k)$ is also a polynomial, but for a massless vector field — or any other kind of a gauge field — it becomes non-polynomial and gauge-dependent. I shall explain the Feynman propagator for the EM fields later in class, probably sometimes in November.