1. Consider a pair of massive charged superfields in SQED background,

\[ \mathcal{L} = \int d^4 \theta (\overline{A} e^{2V} A + \overline{B} e^{-2V} B) + \int d^2 \theta M_{AB} + \int d^2 \theta M^* \overline{A} B + \mathcal{L}(V). \]  

(1)

In the limit of large scalar mass \( M \) and/or slowly varying vector superfield background \( V(z) \), the expectation value of the gauge-invariant chiral operator product \( A(z)B(z) \) becomes

\[ \langle A(z)B(z) \rangle \xrightarrow{M \to \infty} \frac{1}{16\pi^2 M} W^{\alpha}(z) W_{\alpha}(z) + O(M^{-3}). \]  

(2)

In the Superspace book, this formula obtains from a single one-loop diagram involving covariantly-chiral superfields. Unfortunately, deriving the covariantly-chiral superfield formalism and Feynman rules goes beyond the scope of this course, so you should use the ordinary superfield Feynman rules in this exam.

(a) Write down all the one-loop Feynman supergraphs involving two external photons legs and one fixed-in-superspace vertex \( A(z)B(z) \) (no \( \int d^8 z \)). Mind the massive propagators.

(b) Show that the UV divergence of each diagram is no worse than logarithmic — which allows shifting loop momenta and combining the integrands of momentum integrals before the integration. Combine the diagrams and show that they add up to

\[ \frac{M^*}{256} \int d^8 z_1 \int d^8 z_2 \Delta_m(z - z_1) \overline{D}^2_{(1)} \overline{D}^2_{(1)} \cdot V(z_1) \Delta_m(z_1 - z_2) \overline{D}^2_{(2)} \overline{D}^2_{(2)} \Delta_m(z_2 - z) \]

where

\[ \Delta_m(z_1 - z_2) \overset{\text{def}}{=} \frac{-i}{\partial_{(1)}^2 + |m|^2 - 0i} \delta^{(8)}(z_1 - z_2) \]

and ditto for the \( \Delta_m(z - z_1) \) and \( \Delta_m(z_2 - z) \).

(c) Use \( D^\alpha, \overline{D}^\dot{\alpha} \) algebra to evaluate this integral and show that the result agrees with eq. (2).
(d) Now consider diagrams with more (or fewer) external photons. Argue that
\[
\langle A(z)B(z) \rangle = \sum M^{2-d} \mathcal{O}_d(z)
\]
where \( \mathcal{O}_d(z) \) are local operators of dimension \( d \) with appropriate symmetry properties, and that \( W^\alpha W_\alpha \) has the lowest dimension among the allowed operators.

2. Now consider SQED with massless charged fields \( A \) and \( B \). Let
\[
J(z) \overset{\text{def}}{=} \overline{A} e^{+2V} + e^{-2V} \overline{B}.
\] (3)

Classically, \( D^2 J = \overline{D}^2 J = 0 \) by equations of motion for the charged fields, but in the quantum theory
\[
\overline{D}^2 J = \frac{1}{2\pi^2} W^\alpha W_\alpha, \quad D^2 J = \frac{1}{2\pi^2} W_\alpha W^\alpha
\] (4)
because of the one-loop \textit{Konishi} anomaly.

(a) Among other things, the composite superfield \( J \) contains the axial current of the fermionic fields comprising \( A \) and \( B \). Indeed, show that
\[
j^m(x) \overset{\text{def}}{=} \frac{1}{4} \sigma^m_{\beta\dot{\alpha}} \left[ \overline{D}^\beta, D^\dot{\alpha} \right] J(x, \theta, \bar{\theta}) \bigg|_{\theta=\bar{\theta}=0} = \bar{\psi}_A \dot{\sigma}^m \psi_A + \bar{\psi}_B \bar{\sigma}^m \psi_B + \cdots.
\] (5)

(b) Show that Konishi eqs. (4) include the ordinary Adler–Bell–Jackiw axial anomaly
\[
\partial_m j^m = \frac{e}{16\pi^2} F_{kl} F_{mn} + \cdots.
\] (6)

Hint: Evaluate \( [\overline{D}^2, D^2] J \).
In terms of Feynman supergraphs, the Konishi anomaly of the quantum theory is a sum of one-loop graphs with a charged field \((A\) or \(B\)) in the loop, several external photons and one external leg for the \(J\) current or rather its non-conservation \(D^2 J\) (or \(D^2 \bar{J}\)). Generally, such diagrams are quadratically divergent, which requires us to regularize each diagram separately before we may combine them into a single momentum integral. Indeed, the formal sum of the un-regularized diagrams cancels — for the ‘right’ identification of their loop momenta — but the regularized diagrams add up to a finite non-zero anomaly (4).

(c) Verify such formal — but unphysical — cancellation among the diagrams with two external photon legs.

\((c^*)\) For extra credit, show a similar formal cancellation between diagrams with any fixed number of external photon legs.

The simplest way to separately regularize Feynman diagrams of different topologies is Pauli–Villars: Let us add to the theory a pair of un-physical, ultra-heavy chiral superfields \(X\) and \(Y\) of the same charges as \(A\) and \(B\) but of wrong statistics, hence an extra ‘−’ sign for every loop of \(X\) or \(Y\). Such loops have the same topology and same momenta as loops of \(A\) or \(B\) — which allows combining before integration, hence UV regularization.

(d) Show that the regularized Konishi ‘current’

\[
J^\text{reg} = \left( \bar{A}e^{+2V}A + \bar{B}e^{-2V}B \right) + \left( \bar{X}e^{+2V}X + \bar{Y}e^{-2V}Y \right)
\]

satisfies

\[
\overline{D^2 J}^\text{reg} = 8M_{PV}XY,
\]

\[
D^2 J^\text{reg} = 8M_{PV}^*XY,
\]

which leads to eqs. (4) in the limit of infinite Pauli–Villars mass \(M_{PV}\), cf. problem 1.