Consider an $\mathcal{N} = 1$ supersymmetric gauge theory with a simple non-abelian gauge group $G$ and $3 \dim(G)$ chiral superfields $\Phi^k_a$ comprising 3 copies of the adjoint representation of $G$ ($k = 1, 2, 3, a = 1, \ldots, \dim(G)$). In matrix notations, we have a matrix-valued vector superfield $V(x, \theta, \bar{\theta})$ and three matrix-valued chiral superfields $\Phi^k(y, \theta)$ together with their anti-chiral conjugates $\bar{\Phi}^k(y, \bar{\theta})$. The action

$$
S = \int d^4x d^2\theta d^2\bar{\theta} \frac{2}{2} \text{tr} \left( \Phi^k e^{2V} \Phi^k e^{-2V} \right) + \int d^4x d^2\theta \text{tr} \left( \frac{1}{2g^2} \epsilon_{\alpha\beta} W^\alpha W^\beta + \frac{Y}{3} \epsilon_{ijkl} \Phi^i \Phi^j \Phi^k \Phi^l \right) + \int d^4x d^2\bar{\theta} \text{tr} \left( \frac{1}{2g^2} \epsilon_{\dot{\alpha}\dot{\beta}} \dot{W}^\dot{\alpha} \dot{W}^\dot{\beta} + \frac{Y}{3} \epsilon_{ij\ell k} \bar{\Phi}^i \bar{\Phi}^j \bar{\Phi}^k \bar{\Phi}^\ell \right)
$$

has a manifest $SU(3)$ flavor symmetry (acting on the $k$ index of the $\Phi^k$ superfields). In fact it’s the only renormalizable action with this symmetry.

1. Expand the action (1) in terms of the component fields. Use the Wess–Zumino gauge for the vector superfield $V$.

2. Eliminate the auxiliary fields by their equations of motion and derive the scalar potential $V(\varphi^k, \varphi_k^\dagger)$.

3. Show that for a particular value $Y_0$ of the superpotential coupling $Y$, the scalar potential has an $SO(6)$ symmetry. Specifically, in terms of the hermitian scalar fields

$$
H_k = \frac{\varphi^k + \varphi_k^\dagger}{\sqrt{2}}, \quad H_{k+3} = \frac{\varphi^k - \varphi_k^\dagger}{\sqrt{2i}}
$$

the potential becomes

$$
V(s) \propto \sum_{I,J=1,\ldots,6} \text{tr} \left( -[H_I, H_J]^2 \right).
$$

4. Show that for $Y = Y_0$, the whole action of the component-field theory is $\text{Spin}(6) = SU(4)$ invariant. To make this symmetry manifest, you should rescale the fermionic fields so
that all of them are normalized canonically. Assuming $2 \text{tr}(t^a t^b) = \delta^{ab}$ for the generators $t_a$ of the gauge group, the kinetic-energy terms for the fermions should read

$$\mathcal{L} \supset i \bar{\psi}_p^a \sigma^m \nabla_m \psi^{a,p} = 2i \text{tr}(\bar{\psi}_p \sigma^m \nabla_m \psi^p) \quad (4)$$

where $p = 1, 2, 3, 4$ and $\psi^4_\alpha = \lambda_\alpha$ and $\bar{\psi}^4_\dot{\alpha} = \bar{\lambda}_{\dot{\alpha}}$.

5. Show that this particular $SU(4)$ symmetry combined with the $\mathcal{N} = 1$ SUSY implies that the theory must have extended $\mathcal{N} = 4$ SUSY and that the $SU(4)$ is its $R$-symmetry.

6. Evaluate the action of the supercharges $Q_{p,\alpha}$ and $\overline{Q}_\dot{\alpha}^p$ on the component fields of the theory. That is, evaluate $[Q_{p,\alpha}, A^m]$, $[Q_{p,\alpha}, H_I]$, $\{Q_{p,\alpha}, \lambda_q^\beta\}$, $\{Q_{p,\alpha}, \bar{\lambda}_{q,\dot{\beta}}\}$ and ditto for the $\overline{Q}_\dot{\alpha}^p$.

Hint: Identify $Q_{4,\alpha}$ and $\overline{Q}_\dot{4}^\alpha$ as the manifest supercharges of the $\mathcal{N} = 1$ superfield formulation, eliminate the auxiliary fields to obtain on-shell formulæ, then use the $SU(4)$ symmetry.

* For extra credit: Verify that the $\mathcal{N} = 4$ superalgebra closes on-shell — i.e., modulo terms that vanish by field equations of motion — and also modulo gauge transforms of the vector fields $A^m$.

7. Verify that the component-field action is indeed $\mathcal{N} = 4$ supersymmetric.

8. Finally, show that the $\mathcal{N} = 4$ SSYM theory in $d = 4$ dimensions is precisely the dimensional reduction of the $\mathcal{N} = 1$ SSYM theory in $d = 10$ dimensions.

* For extra credit: Write down the supersymmetry action in the $d = 10$ SSYM theory and verify that the SUSY algebra closes on-shell.
Several problem in this exam involve $SU(4) = \text{Spin}(6)$ group theory which may be unfamiliar to some of the students. In this appendix, I explain how to relate the $4$, the $\bar{4}$ and the $6$ representations are related to each other.

From the Spin(6) point of view, $6$ is the vector representation, $4$ is the spinor and the $\bar{4}$ is the other spinor; the two spinors are complex conjugates of each other. From the $SU(4)$ point of view, $4$ is the fundamental representation, $\bar{4}$ is its conjugate, and $6$ is the antisymmetric tensor representation — or the conjugate antisymmetric tensor representation; Translating between the pictures, we may write the $6$ scalar fields of the $N = \triangle$ SSYM theory as $H_I = H_I^\dagger$ or as $H_{[pq]} \equiv C_{I,pq} H_I$ or as $H_I^{[pq]} = C_I^{pq} H_I$. (Please note that each of the $H_{[pq]}$ or $H_I$ is itself a matrix with respect to the gauge group $G$.) The coefficients $C_{I,pq}$ of this translation satisfy several useful equations:

\begin{align*}
C_{I,pq} & = -C_{I,qp} = \epsilon_{pqr}s C_{I}^{rs}, \\
C_{I,pq} C_{J}^{pq} & = 4 \delta_{IJ}, \\
C_{I,pq} C_{I}^{rs} & = 2 (\delta^{r}_{p}\delta^{s}_{q} - \delta^{s}_{p}\delta^{r}_{q}), \\
C_{I,pq} C_{J}^{sr} + C_{J,pq} C_{I}^{sr} & = -2 \delta_{IJ} \delta^{r}_{p}. 
\end{align*}

(5)

In $SU(3) \subset SU(4)$ terms, i.e. distinguishing between $p = k = 1, 2, 3$ and $p = 4$, one has

\begin{align*}
C_{I,k\ell} & = \epsilon_{k\ell I} + i \epsilon_{k\ell(I-3)} , \quad C_{I,k4} = -C_{I,4k} = \delta_{kI} - i\delta_{k(I-3)}, \quad C_{I,44} = 0, 
\end{align*}

(6)

and consequently

\begin{align*}
H_{k\ell} & = \sqrt{2} \epsilon_{k\ell j} \varphi^{j}, \quad H_{k4} = -H_{4k} = \sqrt{2} \varphi_{k}^{*}, \quad H_{44} = 0.
\end{align*}
You may also find useful a representation of the $d = 9+1$ Dirac $\Gamma_M$ matrices in $SO(3 + 1) \times SO(6)$ terms. First, we define $16 \times 16$ Weyl matrices
\[
\Sigma_m = \begin{pmatrix}
\sigma_m \otimes 1_{4 \times 4} & 0 \\
0 & \bar{\sigma}_m \otimes 1_{4 \times 4}
\end{pmatrix}, \quad \Sigma_m = \begin{pmatrix}
\sigma_m \otimes 1_{4 \times 4} & 0 \\
0 & \bar{\sigma}_m \otimes 1_{4 \times 4}
\end{pmatrix}
\]
for $m = 0, 1, 2, 3$, and
\[
\Sigma_{(I+3)} = -\Sigma_{(I+3)} = \begin{pmatrix}
0 & 1_{2 \times 2} \otimes C_I \\
1_{2 \times 2} \otimes C_I^\dagger & 0
\end{pmatrix}
\]
for $I + 3 = 4, \ldots, 9$. Note that similar to their $d = 4$ counterparts, all these matrices are hermitian, $\Sigma_0 = \Sigma_0 = 1$ while $\Sigma_M = -\Sigma_M$ for $M \neq 0$, and
\[
\Sigma_M \Sigma_N + \Sigma_N \Sigma_M = 2g_{MN} = \Sigma_M \Sigma_N + \Sigma_N \Sigma_M.
\]
Another useful matrix is
\[
\Xi = \Sigma_2 \Sigma_7 \Sigma_8 \Sigma_9 = \Sigma_2 \Sigma_7 \Sigma_8 \Sigma_9 = \begin{pmatrix}
0 & \epsilon \otimes 1_{4 \times 4} \\
-\epsilon \otimes 1_{4 \times 4} & 0
\end{pmatrix}
\]
which satisfies $\Sigma_C \Sigma_M = \Sigma_M^* \Sigma_C$.

Consequently, we define the $32 \times 32$ Dirac matrices according to
\[
\Gamma_M = \begin{pmatrix}
0 & \Sigma_M \\
\Sigma_M & 0
\end{pmatrix}
\]
and have them satisfy the usual anticommutation relations $\Gamma_M \Gamma_N + \Gamma_N \Gamma_M = 2g_{MN}$. In the basis (7), Majorana–Weyl fermions satisfy
\[
\Psi = +\Gamma_{11} \Psi = +\Gamma_C \Psi^*
\]
where
\[
\Gamma_{11} = \begin{pmatrix}
+1 & 0 \\
0 & -1
\end{pmatrix} \quad \text{and} \quad \Gamma_C = \Gamma_2 \Gamma_7 \Gamma_8 \Gamma_9 = \begin{pmatrix}
\Xi & 0 \\
0 & \Xi
\end{pmatrix}.
\]