Mandelstam Variables

Consider any kind of a $2$ particles $\rightarrow 2$ particles process

The 4-momenta $p_1^\mu$, $p_2^\mu$, $p_1'^\mu$, and $p_2'^\mu$ of the 2 incoming and 2 outgoing particles are on-shell satisfy 8 constraints: the on-shell conditions for each particle

$$p_1^2 = m_1^2, \quad p_2^2 = m_2^2, \quad p_1'^2 = m_1'^2, \quad p_2'^2 = m_2'^2$$

and the net 4-momentum conservation

$$p_1 + p_2 - p_1' - p_2' = 0.$$ 

Consequently, all Lorentz invariant combinations of the 4 external momenta may be expressed in terms of the particles’ masses and 3 Mandelstam’s variables

$$s = (p_1 + p_2)^2 = (p_1^1 + p_2^1)^2,$$

$$t = (p_1 - p_1')^2 = (p_2^2 - p_2)^2,$$

$$u = (p_1 - p_2')^2 = (p_1^1 - p_2^2)^2.$$ 

Moreover, only 2 out of these 3 variables are independent while their sum has a fixed value

$$s + t + u = m_1^2 + m_2^2 + m_1'^2 + m_2'^2.$$ 

Indeed,

$$s + t + u = (p_1 + p_2)^2 + (p_1 - p_1')^2 + (p_1 - p_2')^2$$

$$= 3p_1^2 + p_2^2 + p_1'^2 + p_2'^2 + 2(p_1p_2) - 2(p_1p_1') - 2(p_1p_2')$$

$$= p_1^2 + p_2^2 + p_1'^2 + p_2'^2 + 2p_1 \times (p_1 + p_2 - p_1' - p_2' = 0)$$

$$= m_1^2 + m_2^2 + m_1'^2 + m_2'^2.$$ 

In terms of Mandelstam’s variables, Lorentz products of momenta are given by

\[ 2(p_1p_2) = (p_1 + p_2)^2 - p_1^2 - p_2^2 = s - m_1^2 - m_2^2, \]
\[ 2(p'_1p'_2) = (p'_1 + p'_2)^2 - p'_1^2 - p'_2^2 = s - m_1^2 - m_2^2, \]
\[ 2(p_1p'_1) = p_1^2 + p_1'^2 - (p_1 - p'_1)^2 = m_1^2 + m_1'^2 - t, \]
\[ 2(p_2p'_2) = p_2^2 + p_2'^2 - (p_2 - p'_2)^2 = m_2^2 + m_2'^2 - t, \]
\[ 2(p_1p'_2) = p_1^2 + p_2'^2 - (p_1 - p'_2)^2 = m_1^2 + m_2'^2 - u, \]
\[ 2(p_2p'_1) = p_2^2 + p_1'^2 - (p_2 - p'_1)^2 = m_2^2 + m_1'^2 - u. \]  

In particular, for an elastic scattering of 2 same-mass particles

\[ s + t + u = 4m^2, \]
\[ 2(p_1p_2) = 2(p'_1p'_2) = s - 2m^2, \]
\[ 2(p_1p'_1) = 2(p_2p'_2) = 2m^2 - t, \]
\[ 2(p_1p'_2) = 2(p_2p'_1) = 2m^2 - u. \]  

For the future reference, let me give you similar formulae for the \( e^-e^+ \rightarrow \mu^-\mu^+ \) pair-production,

\[ s + t + u = 2M_{\mu}^2 + 2m_e^2 \approx 2M_{\mu}^2, \]
\[ 2(p_1p_2) = s - 2m_e^2 \approx s, \]
\[ 2(p'_1p'_2) = s - 2M_{\mu}^2, \]
\[ 2(p_1p'_1) = 2(p_2p'_2) = M_{\mu}^2 + m_e^2 - t \approx M_{\mu}^2 - t, \]
\[ 2(p_1p'_2) = 2(p_2p'_1) = M_{\mu}^2 + m_e^2 - u \approx M_{\mu}^2 - u. \]  

and for the \( e^-e^+ \rightarrow \gamma\gamma \) annihilation process \( p_- + p_+ \rightarrow k_1 + k_2 \),

\[ s + t + u = 2m_e^2, \]
\[ 2(p_-p_+) = s - 2m_e^2, \]
\[ 2(k_1k_2) = s, \]
\[ 2(p_-k_1) = 2(p_+k_2) = m_e^2 - t, \]
\[ 2(p_-k_2) = 2(p_+k_1) = m_e^2 - u. \]