QED Feynman rules

Quantum Electro Dynamics or QED is the theory of EM field $A_\mu(x)$ coupled to the electron field $\Psi(x)$ (and optionally other charged fermion fields). The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\Psi} (i \gamma^\mu - m) \Psi$$

$$= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\Psi} (i \gamma^\mu - m) \Psi + e A_\mu \times \overline{\Psi} \gamma^\mu \Psi$$

(1)

where the first 2 terms on the last line describe free photons and electrons $e^\pm$, and the third term is treated as a perturbation.

The two different field types have different propagators

$$S_{\alpha\beta}^{F}(x-y) = \langle 0 | T \Psi_{\alpha}(x) \overline{\Psi}_{\beta}(y) | 0 \rangle$$

and

$$G_{F}^{\mu\nu}(x-y) = \langle 0 | T A^\mu(x) A^\nu(y) | 0 \rangle$$

In QED Feynman rules, these propagators are denoted by two different types of internal lines: The electron propagator is drawn as a solid line with an arrow indicating which end of the line belongs to a $\Psi$ field and which to a $\overline{\Psi}$,

$$\overline{\Psi}_\alpha \rightarrow \Psi_\beta \quad = \quad \begin{bmatrix} i \frac{1}{q - m + i0} \end{bmatrix}_{\alpha\beta}.$$  

(2)

The smaller arrow near $q$ indicates the direction of the momentum flow. Both arrows should have the same direction; otherwise we would have

$$\overline{\Psi}_\alpha \rightarrow \Psi_\beta \quad = \quad \begin{bmatrix} i \frac{1}{-q - m + i0} \end{bmatrix}_{\alpha\beta}.$$  

(3)

The photon propagator is drawn as a wavy line without arrow,

$$A^\mu \quad \rightarrow \quad A^\nu \quad = \quad \frac{i C^{\mu\nu}(q)}{q^2 + i0}$$

(4)

where the $C^{\mu\nu}(q)$ factor in the numerator depends on the gauge-fixing condition for the EM fields. Most generally

$$C^{\mu\nu}(q) = -g^{\mu\nu} + q^\mu \times t^\nu(q) + q^\nu \times t^\mu(q)$$

(5)

for some $q$-dependent vector $t^\mu(q)$. For example, in the Coulomb gauge $\nabla \cdot A \equiv 0$,

$$t^\mu(q) = \frac{(q^0, -q)}{2q^2} \quad \Rightarrow \quad C^{00} = \frac{(q^0)^2 - q^2}{q^2}, \quad C^{0i} = C^{i0} = 0, \quad C^{ij} = \delta^{ij} - \frac{q^i q^j}{q^2},$$

(6)
Thus

\[ A^i \bullet A^j \quad q \rightarrow \quad i \frac{q^i q^j}{q^2 + i0} \times (\delta^{ij} - \frac{q^i q^j}{q^2}) \quad \text{while} \quad A^0 \bullet A^0 \quad q \rightarrow \quad i \frac{q^0}{q^2} . \quad (7) \]

A Lorentz-invariant gauge condition \( \partial_\mu A^\mu(x) \equiv 0 \) — called the Landau gauge — leads to \( t^\mu = q^\mu/(2q^2) \) and hence Lorentz-symmetric photon propagator

\[ A^\mu \bullet A^\nu \quad q \rightarrow \quad \frac{-i}{q^2 + i0} \times \left( g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2 + i0} \right) . \quad (8) \]

There are many other gauges with different \( t^\mu(q) \), but fortunately, when the photon is coupled to conserved electric currents, the \( q^\mu t^\nu + t^\mu q^\nu \) terms in the propagator’s numerator become irrelevant because on each side of the propagator

\[ q^\mu J^{(1)}_{\mu}(q) = q^\nu \times J^{(2)}_{\nu}(q) = 0 \quad \implies \quad J^{(1)}_{\mu}(q) \times C^{\mu\nu}(q) \times J^{(2)}_{\nu}(q) = J^{(1)}_{\mu}(q) \times (-g^{\mu\nu}) \times J^{(2)}_{\nu}(q) . \quad (9) \]

To be precise, the gauge-dependent terms \( q^\mu t^\nu + t^\mu q^\nu \) may contribute to some individual Feynman diagrams, but once we sum over all diagrams contributing to the same physical QED amplitude, the gauge-dependence always cancels out. But to make sure this works, we must use the same gauge for all the propagators in all the contributing diagrams.

In this class we shall use the Feynman gauge where \( t^\nu \equiv 0 \) and the propagator is simply

\[ A^\mu \bullet A^\nu \quad q \rightarrow \quad \frac{-i g^{\mu\nu}}{q^2 + i0} . \quad (10) \]

Defining the Feynman gauge in terms of restrictions on the \( A^\mu(x) \) fields is rather complicated, so I’ll postpone this issue until second half of the Spring semester; all we need for now is the photon propagator (10).

The vertices of Feynman diagrams follow from the interaction terms in the Lagrangian that involve 3 or more fields. The QED Lagrangian has only one interaction term \( eA_\mu \times \bar{\Psi} \gamma^\mu \Psi \), so there is only one vertex type, namely

\[ \beta \]

\[ \mu \]

\[ \alpha \]

\( = ( +ie\gamma^\mu )_{\beta\alpha} . \quad (11) \]

This vertex has valence = 3, and the 3 lines it connects must be of specific types: one wavy (photonic) line, one solid line with incoming arrow, and one solid line with outgoing arrow.
Now consider the external lines. When a quantum EM field

\[
\hat{A}^\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \sum_{\lambda=\pm 1} \left( e^{-ikx} \epsilon^\mu(k, \lambda) \times \hat{a}_{k,\lambda} + e^{+ikx} \epsilon^{\mu*}(k, \lambda) \times \hat{a}^{\dagger}_{k,\lambda} \right) k^0 = +\omega_k
\]  

(12)

is contracted with an incoming or an outgoing photon, we end up with an external-line factor accompanying the matching \( \hat{a}_{k,\lambda} \) or \( \hat{a}^{\dagger}_{k',\lambda'} \) operator, namely \( e^{-ikx} \times \epsilon^\mu(k, \lambda) \) for an incoming photon or \( e^{+ik'x} \times \epsilon^{\mu*}(k', \lambda') \) for an outgoing photon. The momentum space Feynman rules take care of the \( e^{\pm ikx} \) factors, which leaves us with the polarization vectors \( \epsilon^\mu(k, \lambda) \) or \( \epsilon^{\mu*}(k', \lambda') \) for the external photon lines:

\[
\begin{align*}
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \}
The QED vertex (11) has one incoming fermionic line and one outgoing, and we may think of them as being two segments of single continuous line going through the vertex. From this point of view, a fermionic line enters a diagram as an incoming $e^-$ or an outgoing $e^+$, goes through a sequence of vertices and propagators, and eventually exits the diagram as an outgoing $e^-$ or and incoming $e^+$,

$$
\begin{array}{c}
e^- & \text{in} \\
e^+ & \text{out}
\end{array}
\end{array}
\quad
\begin{array}{c}
e^- & \text{out} \\
e^+ & \text{in}
\end{array}
$$

(The photonic lines here may be external or internal; if internal, they connect to some other fermionic lines, or maybe even to the same line at another vertex.) Alternatively, a fermionic line may form a closed loop, like

The continuous fermionic lines such as (19) or (20) are convenient for handling the Dirac indices of vertices, fermionic propagators, and external lines. For an open line such as (19), the rule is to read the line in order, from its beginning to its end, spell all the vertices, the propagators, and the external line factors in the same order right-to-left, then multiply them together as Dirac matrices. For example, consider a diagram where an incoming electron and incoming positron annihilate into 3 photons, real or virtual. This diagram has a fermionic line which starts at the incoming $e^-$, goes through 3 vertices and 2 propagators, and exits at the incoming $e^+$ as shown below:

$$
\begin{array}{c}
\nu \\
\eta
\end{array}
\quad
\begin{array}{c}
\mu \\
\eta
\end{array}
\quad
\begin{array}{c}
\lambda \\
\eta
\end{array}
\end{array}
\quad
\begin{array}{c}
in e^+(p_+, s_+) \\
q_2 \\
q_1 \\
in e^-(p_-, s_-)
\end{array}
$$

The propagators here carry momenta $q_1 = p_- - k_1$ and $q_2 = q_1 - k_2 = k_3 - p_+$. The fermionic line (21) carries the following factors:
• $u(p_-, s_-)$ for the incoming $e^-$;
• $+ie\gamma^\lambda$ for the first vertex (from the right);
• $\frac{i}{\not{q}_1 - m + i\epsilon}$ for the first propagator;
• $+ie\gamma^\mu$ for the second vertex;
• $\frac{i}{\not{q}_2 - m + i\epsilon}$ for the second propagator;
• $+ie\gamma^\nu$ for the third vertex;
• $\bar{v}(p_+, s_+)$ for the incoming $e^+$.

Reading all these factors in the order of the line (21), tail-to-head, and multiplying them right-to-left, we get the following Dirac ‘sandwich’

$$
\bar{v}(p_+, s_+) \times (+e\gamma^\nu) \times \frac{i}{\not{q}_2 - m + i\epsilon} \times (+ie\gamma^\mu) \times \frac{i}{\not{q}_1 - m + i\epsilon} \times (+ie\gamma^\lambda) \times u(p_-, s_-).
$$

This formula, all the Dirac indices are suppressed; the rule is to multiply all factors as Dirac matrices (or row / column spinors) in this order.

For a closed fermionic loop such as (20), the rule is to start at an arbitrary vertex or propagators, follow the line until one gets back to the starting point, multiply all the vertices and the propagators right-to-left in the order of the line, then take the trace of the matrix product. For example, the loop

\begin{tikzpicture}[baseline=(current bounding box.center)]
  \draw (0,0) -- (1,1);
  \draw (1,1) -- (2,0);
  \draw (2,0) -- (3,1);
  \draw (3,1) -- (0,0);

  \draw (0.5,0.5) node {$q_1$};
  \draw (1.5,1.5) node {$q_2$};
  \draw (2.5,0.5) node {$q_3$};
  \draw (1.5,0.5) node {$q_4$};

  \draw (0.25,0.25) node {$\kappa$};
  \draw (1.75,0.75) node {$\lambda$};
  \draw (0.75,2.75) node {$\nu$};
  \draw (2.25,0.25) node {$\mu$};
\end{tikzpicture}

produces Dirac trace

$$
\text{tr}\left[ (+ie\gamma^\kappa) \times \frac{i}{\not{q}_1 - m + i\epsilon} \times (+ie\gamma^\lambda) \times \frac{i}{\not{q}_2 - m + i\epsilon} \times (+ie\gamma^\mu) \times \frac{i}{\not{q}_3 - m + i\epsilon} \times (+ie\gamma^\nu) \times \frac{i}{\not{q}_4 - m + i\epsilon} \right].
$$

Note that a trace of a matrix product depends only on the cyclic order of the matrices,
(tr(ABC \cdots YZ) = tr(BC \cdots YZA) = tr(C \cdots YZAB) = \cdots = tr(ZABC \cdots Y)). Thus, in eq. (24), we may start the product with any vertex or propagator — as long as we multiply them all in the correct cyclic order, the trace will be the same.

As to the Lorentz vector indices \( \lambda, \mu, \nu, \ldots \), the index of a vertex should be contracted to the index of the photonic line connected to that vertex. For example, the following diagram for \( e^- + e^- \rightarrow e^- + e^- \) scattering

![Diagram](image)

(25)
evaluates to

\[
iM = (\bar{u}(p'_1, s'_1) \times (\pm ie\gamma_\mu) \times u(p_1, s_1)) \times (\bar{u}(p'_2, s'_2) \times (\pm ie\gamma_\nu) \times u(p_2, s_2)) \times \frac{-ig^{\mu\nu}}{q^2}. \quad (26)
\]

Here we have used the Feynman gauge for the photon propagator, but any other gauge would produce exactly the same amplitude

\[
iM = \bar{u}'_1(i\epsilon\gamma_\mu)u_1 \times \bar{u}'_2(i\epsilon\gamma_\nu)u_2 \times \frac{i(-g^{\mu\nu} + t^\mu q^\nu + q^\mu t^\nu)}{q^2} \quad (27)
\]

because of Gordon identities

\[
\bar{u}'_1(i\epsilon\gamma_\mu)u_1 \times q^\mu = \bar{u}'_2(i\epsilon\gamma_\nu)u_2 \times q^\nu = 0. \quad (28)
\]

To prove these identities, we note that the spinors \( u_1 \equiv u(p_1, s_1) \) and \( \bar{u}'_1 \equiv \bar{u}(p'_1, s'_1) \) satisfy Dirac equation

\[
p_1' u_1 = m u_1, \quad \bar{u}'_1 p'_1 = m \bar{u}'_1. \quad (29)
\]

Moreover, \( q = p'_1 - p_1 \) and hence

\[
\bar{u}'_1 \gamma_\mu u_1 \times q^\mu = \bar{u}'_1 \not{q} u_1 = \bar{u}'_1(p'_1 - p_1)u_1 = (m \bar{u}'_1)u_1 - \bar{u}'_1(m u_1) = 0. \quad (30)
\]
Similarly, \( q = p_2 - p_2' \) and hence

\[
\bar{u}_2' \gamma_\mu u_2 \times q^\mu = \bar{u}_2' \not{q} u_2 = \bar{u}_2'(p_2 - p_2')u_1 = \bar{u}_2'(mu_2)u_2 - (m\bar{u}_2') = 0. \tag{31}
\]

For other combinations of incoming or outgoing electrons or positrons connected to the same vertex we have similar Gordon identities:

\[
\begin{align*}
\bar{v}(p) \not{q} v(p') &= 0 \quad \text{for } q = p' - p, \\
\bar{v}(p_2) \not{q} u(p_1) &= 0 \quad \text{for } q = p_1 + p_2, \\
\bar{u}(p_2') \not{q} v(p_2') &= 0 \quad \text{for } q = p_1' + p_2',
\end{align*}
\tag{32}
\]

which provide for gauge invariance of Feynman diagrams like

\[
\begin{align*}
\text{or}
\end{align*}
\tag{33}
\]

In general, an individual Feynman diagram is not always gauge-independent. However, when one sums over all diagrams contributing to some scattering process at some order, the sum is always gauge invariant. We shall return to this issue later this semester.

To complete the QED Feynman rules, we need to keep track of the ‘\(-\)’ signs arising from re-ordering of fermionic fields and creation / annihilation operators. To save time, I will not go through the gory details of the perturbation theory. Instead, let me simply state the rules for the overall sign of a Feynman diagram in terms of the continuous fermionic lines:

- There is a ‘\(-\)’ sign for every closed fermionic loop.
- There is a ‘\(-\)’ sign for every open fermionic line which begins at an outgoing positron and ends at an incoming positron.
- There is a ‘\(-\)’ sign for every crossing of the fermionic lines.

Although the number of such crossing depends on how we draw the diagram on a 2D
sheet of paper, for example

\[ \begin{array}{c}
1' \\
\downarrow \\
1 \\
\uparrow \\
2 \\
\downarrow \\
2'
\end{array} \quad \text{versus} \quad \begin{array}{c}
1' \\
\downarrow \\
1 \\
\uparrow \\
2 \\
\downarrow \\
2'
\end{array} \] (34)

but the \#crossings \textbf{mod} 2 is a topological invariant, and that’s all we need to determine the overall sign of the diagram.

⋆ If multiple Feynman diagrams contribute to the same process, then the external legs should stick out from the diagram in the same order for all the diagrams. Or at least all the fermionic external legs should stick out in the same order, which should also agree with the order of fermions in the bra and ket states of the S–matrix element

\[ \langle e^-, \ldots, e^-, e^+, \ldots, e^+, \gamma', \ldots, \gamma' | M | e^-, \ldots, e^-, e^+, \ldots, e^+, \gamma, \ldots, \gamma \rangle \] for the process in question.

Finally, the QED is usually extended to include other charged fermions besides \( e^\mp \). The simplest extension includes the muons \( \mu^\mp \) and the tau leptons \( \tau^\mp \) which behave exactly like the electrons, except for larger masses: while \( m_e = 0.51100 \text{ MeV} \), \( m_\mu = 105.66 \text{ MeV} \) and \( m_\tau = 1777 \text{ MeV} \). In terms of the Feynman rules, the muons and the taus have exactly the same vertices, propagators, or external line as the electrons, except for a different mass \( m \) in the propagators. To distinguish between the 3 lepton species, one should label the solid lines with \( e, \mu, \text{ or } \tau \). Different species do not mix, so a label belongs to the whole continuous fermionic line; for an open line, the species must agree with the incoming / outgoing particles at the ends of the line; for a closed loop, one should sum over the species \( \ell = e, \mu, \tau \).

\textbf{Coulomb Scattering}

As an example of QED Feynman rules, consider the elastic scattering of two electrons,
\( e^- + e^- \rightarrow e^- + e^- \). There are two tree diagrams contributing to this process, namely
\[
\begin{array}{c}
\begin{array}{c}
\text{\(1'\)} \\
\text{\(1\)}
\end{array}
\begin{array}{c}
\text{\(q\)}
\end{array}
\begin{array}{c}
\text{\(2'\)} \\
\text{\(2\)}
\end{array}
\end{array}
+ \begin{array}{c}
\begin{array}{c}
\text{\(1'\)} \\
\text{\(1\)}
\end{array}
\begin{array}{c}
\text{\(q\)}
\end{array}
\begin{array}{c}
\text{\(2'\)} \\
\text{\(2\)}
\end{array}
\end{array}
\tag{35}
\end{array}
\]
these two diagrams are related by exchanging the final-state electrons, \(1' \leftrightarrow 2'\). The first diagram (35) was evaluated back in eq. (26) as
\[
iM_1 = \frac{-ig_{\mu\nu}}{q^2 = t} \times \bar{u}_1'(ie\gamma_\mu)u_1 \times \bar{u}_2'(ie\gamma_\nu)u_2. \tag{36}
\]
For the second diagram, we exchange \(u_1' \leftrightarrow u_2'\), change the momentum of the virtual photon from \(q = p_1' - p_1\) to \(\tilde{q} = p_2' - p_1\) and hence \(q^2 = t\) in the denominator to \(\tilde{q}^2 = u\), and there is an overall minus sign due to electron line crossing, thus
\[
iM_2 = -\frac{-ig_{\mu\nu}}{\tilde{q}^2 = u} \times \bar{u}_2'(ie\gamma_\mu)u_1 \times \bar{u}_1'(ie\gamma_\nu)u_2. \tag{37}
\]
Combining the two diagrams, we obtain the net tree-level scattering amplitude as
\[
M_{\text{tree}} = M_1 + M_2 = \frac{e^2}{t} \times \bar{u}_1'\gamma_\mu u_1 \times \bar{u}_2'\gamma_\mu u_2 - \frac{e^2}{u} \times \bar{u}_2'\gamma_\mu u_1 \times \bar{u}_1'\gamma_\mu u_2. \tag{38}
\]
Now let’s take the non-relativistic limit of this scattering amplitude. A non-relativistic electron with 3-momentum \(|p| \ll m\) and energy \(p^0 \approx m\) has plane-wave Dirac spinor
\[
u(p, s) \approx \left( \begin{array}{c} \sqrt{m}\xi \\ \sqrt{m}\xi \end{array} \right) + O(|p|/\sqrt{m}). \tag{39}
\]
Consequently, the Dirac sandwiches \(\bar{u}'\gamma\mu u\) between non-relativistic electron spinors are approximately
\[
\bar{u}(p', s')\gamma^0 u(p, s) = u^\dagger(p', s')u(p, s) \approx 2m \times \xi_{s'}^\dagger\xi_s = 2m \times \delta_{s,s'},
\]
\[
\bar{u}(p', s')\gamma^\mu u(p, s) = u^\dagger(p', s') \begin{pmatrix} +\vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix} u(p, s) = O(p, p') \ll m. \tag{40}
\]
so the scattering amplitude (38) is dominated by the $\mu = 0$ terms. Specifically,

$$M_{\text{tree}} \approx \frac{e^2}{t} \times 2m\delta_{s_1,s_1} \times 2m\delta_{s_2,s_2} - \frac{e^2}{u} \times 2m\delta_{s'_2,s_1} \times 2m\delta_{s'_1,s_2},$$

where on the second line I have used $t \approx - (\mathbf{q} = \mathbf{p}_1' - \mathbf{p}_1)^2$ and $u \approx - (\mathbf{q} = \mathbf{p}_2' - \mathbf{p}_1)^2$ for the non-relativistic electrons (since $q_0 = E_1' - E_1 = O(p^2/m) \ll |\mathbf{q}|$ and likewise $\tilde{q}_0 \ll |\tilde{\mathbf{q}}|$).

Note that despite the non-relativistic limit, the amplitude (41) is relativistically normalized.

In terms of a non-relativistically normalized scattering amplitude

$$f = \frac{M}{8\pi E_{\text{cm}}},$$

we have $E_{\text{cm}} \approx 2m$ and

$$f_{\text{tree}} \approx -\frac{me^2}{4\pi q^2} \times \delta_{s_1',s_1} \delta_{s_2',s_2} + \frac{me^2}{4\pi \tilde{q}^2} \times \delta_{s_2',s_1} \delta_{s_1',s_2}. \quad (43)$$

Now let’s compare the QED amplitude (43) to the non-relativistic amplitude in potential scattering. In the non-relativistic limit, the perturbative expansion of QED in powers of $e^2$ roughly corresponds to the Born series in potential scattering, so the tree-level amplitude (41) should be compared to the first Born approximation

$$f_B(\mathbf{p} \to \mathbf{p}') = -\frac{M_{\text{red}}}{2\pi} \times \tilde{V}(\mathbf{q} = \mathbf{p}_1' - \mathbf{p}_1) = -\frac{M_{\text{red}}}{2\pi} \times \int d^3x_{\text{rel}} V(x_{\text{rel}}) e^{-i\mathbf{q} \cdot x_{\text{rel}}}. \quad (44)$$

Or rather, this is the Born amplitude for distinct spinless particles with a reduced mass $M_{\text{red}}$. For distinct particles with spin interacting via spin-blind potential $V(x_1 - x_2)$, the Born amplitude is

$$f_B = -\frac{M_{\text{red}}}{2\pi} \times \tilde{V}(\mathbf{q}) \times \delta_{s_1',s_1} \delta_{s_2',s_2},$$

and for identical fermions there two such terms due to particle permutations,

$$f_B = -\frac{M_{\text{red}}}{2\pi} \times \tilde{V}(\mathbf{q}) \times \delta_{s_1',s_1} \delta_{s_2',s_2} + \frac{M_{\text{red}}}{2\pi} \times \tilde{V}(\tilde{\mathbf{q}}) \times \delta_{s_2',s_1} \delta_{s_1',s_2}. \quad (46)$$

Comparing this Born amplitude to the non-relativistic limit of the tree-level QED amplitude (43), we immediately see that the QED amplitude is a special case of the Born amplitude.
for

\[ \widetilde{V}(q) = + \frac{e^2}{q^2}. \] (47)

Fourier transforming this formula back to the coordinate space gives us the good old Coulomb potential for the two electrons,

\[ V(x_1 - x_2) = \int \frac{d^3q}{(2\pi)^3} e^{i(x_1 - x_2) \cdot q} \times \frac{+e^2}{q^2} = \frac{+e^2}{4\pi |x_1 - x_2|}. \] (48)

Now consider the electron-positron elastic scattering \( e^- + e^+ \rightarrow e^- + e^+ \) and its non-relativistic limit. Again, there are two tree diagrams contributing to this process

which evaluate to

\[ iM_{\text{tree}} = -\bar{u}'_1(i\gamma_\mu) u_1 \times \bar{v}_2(i\gamma_\nu)v'_2 \times -\frac{ig^{\mu\nu}}{t} + \bar{v}_2(i\gamma_\mu) u_1 \times \bar{u}'_1(i\gamma_\nu)v'_2 \times -\frac{ig^{\mu\nu}}{s}. \] (50)

The overall minus sign of the first term here is due to the outgoing-positron-to-incoming-positron line in the first diagram; in the second diagram, the incoming electron-to-incoming-positron or the outgoing-positron-to-outgoing-electron lines do not carry minus signs. This time, there is no symmetry between the two diagrams (49), and the corresponding amplitudes have rather different non-relativistic limits. In particular, the denominator of the first diagram becomes \( t \approx -q^2 \ll m^2 \) (in absolute value) while the second diagram’s denominator has a much larger value \( s \approx (2m)^2 \). Consequently, the non-relativistic electron-positron scattering is dominated by the t-channel diagram, thus

\[ \mathcal{M}_{\text{tree}}^{\text{non.rel.}} \approx -\frac{e^2}{t} \times \bar{u}'_1\gamma_\mu u_1 \times \bar{v}_2\gamma^\mu v'_2. \] (51)
Moreover, in the non-relativistic limit
\[ \bar{u}'_1 \gamma^0 u_1 \approx +2m \times \delta_{s'_1,s_1}, \quad \bar{v}'_2 \gamma^0 v_2 \approx +2m \times \delta_{s'_2,s_2}, \] (52)

while for \( \mu \neq 0 \)
\[ \bar{u}'_1 \gamma u_1, \quad \bar{v}'_2 \gamma v_2 = O(\mathbf{p}) \ll m, \] (53)
hence
\[ M_{\text{tree}}^{\text{non-rel.}} \approx -\frac{e^2}{t} \times 2m \delta_{s'_1,s_1} \times 2m \delta_{s'_2,s_2} \approx +\frac{4m^2e^2}{q^2} \times \delta_{s'_1,s_1} \delta_{s'_2,s_2}, \] (54)
or in the non-relativistic normalization
\[ f_{\text{tree}}^{\text{non-rel.}} \approx +\frac{me^2}{4\pi q^2} \times \delta_{s'_1,s_1} \delta_{s'_2,s_2}. \] (55)

Comparing this QED amplitude to the Born amplitude (45) for distinct fermions (since an \( e^- \) is distinct from an \( e^+ \)), we see that they agree for
\[ \tilde{V}(q) = -\frac{e^2}{q^2}. \] (56)

In coordinate space terms, this means the attractive Coulomb potential
\[ V(x_1 - x_2) = -\frac{e^2}{4\pi |x_1 - x_2|}. \] (57)

Thus QED perturbation theory confirms the oldest law of electrostatics: the like-sign charges repel, while the unlike-sign charges attract.

However, this rule works only for the forces arising from exchanges of virtual odd spin bosons — such as photons. The forces arising from exchanges of virtual even spin bosons — such as scalar mesons, or gravitons — do not change sign when one of the two particles is replaced with its anti-particle. Thus, the gravity force is always attractive. Likewise, the Yukawa force due to an isoscalar scalar meson is attractive for all combinations of nucleons and antinucleons — \( NN, \overline{NN}, \) or \( N\overline{N} \).
Yukawa Potential

The Yukawa theory and the Yukawa potential are discussed in detail in §4.7 of the Peskin and Schroeder textbook, so in these notes let me simply highlight the differences between the Yukawa theory and the QED. Instead of the EM field, the Yukawa theory has a scalar field $\phi$, thus

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 + \overline{\psi} (i \gamma^\mu - M) \psi - g \phi \times \overline{\psi} \psi.$$  

(For simplicity, I assume a single fermion species.) The Feynman rules of the Yukawa theory have the same fermionic propagators, external line factors, and sign rules as QED, but instead of photon propagators it has scalar propagators — drawn as dotted lines

$$\phi \quad \cdots \cdots \quad \phi \quad q \rightarrow \quad \phi$$

and instead of the fermion-antifermion-photon vertices of QED the Yukawa theory has fermion-antifermion-scalar vertices

$$\beta$$

$$\cdots \cdots \quad \alpha$$

$$= -ig \delta_{\beta \alpha}.$$ 

without the $\gamma^\mu$ matrices.

Consequently, evaluating the scalar analogues of the diagrams (35) and (49) for the $ff \rightarrow ff$ and $f \bar{f} \rightarrow f \bar{f}$ scattering processes, we obtain

$$\mathcal{M}(ff \rightarrow ff) = -\frac{g^2}{t - m^2} \times \bar{u}_1 u_1 \times \bar{u}_2 u_2 + \frac{g^2}{u - m^2} \times \bar{u}_2 u_1 \times \bar{u}_1 u_2.$$  

$$\mathcal{M}(f \bar{f} \rightarrow f \bar{f}) = +\frac{g^2}{t - m^2} \times \bar{u}_1 u_1 \times \bar{v}_2 v_2' - \frac{g^2}{s - m^2} \times \bar{v}_2 u_1 \times \bar{u}_2 v_2'.$$  

In the non-relativistic limit

$$\bar{u}(p', s') u(p, s) \approx +2M \delta_{s',s} \quad \text{but} \quad \bar{v}(p, s) v(p', s') \approx -2M \delta_{s',s},$$  

while

$$\bar{v}(p_2, s_2) u(p_1, s_1), \quad \bar{u}(p'_1, s'_1) v(p'_2, s'_2) = O(p) \ll M,$$
hence
\[ \mathcal{M}(f f \rightarrow f f) \approx -\frac{4M^2g^2}{t - m^2} \times \delta_{s',s_1} \delta_{s',s_2} + \frac{4M^2g^2}{u - m^2} \times \delta_{s',s_1} \delta_{s',s_2}, \]
\[ \mathcal{M}(f \bar{f} \rightarrow f \bar{f}) \approx -\frac{4M^2g^2}{t - m^2} \times \delta_{s',s_1} \delta_{s',s_2} + \frac{g^2}{s - m^2} \times O(p^2). \] (64)

Assuming the scalar is much lighter than the fermion, \( m \ll M \), and taking the fermions’ 3-momenta \( p, p' \) to be \( O(m) \ll M \), we have
\[ \frac{1}{t - m^2} \approx \frac{1}{q^2 + m^2}, \quad \frac{1}{u - m^2} \approx \frac{1}{q^2 + m^2}, \quad \text{but} \quad \frac{1}{s - m^2} \approx \frac{1}{4M^2}, \] (65)

hence
\[ \mathcal{M}(f f \rightarrow f f) \approx +\frac{4M^2g^2}{q^2 + m^2} \times \delta_{s',s_1} \delta_{s',s_2} - \frac{4M^2g^2}{q^2 + m^2} \times \delta_{s',s_1} \delta_{s',s_2}, \]
\[ \mathcal{M}(f \bar{f} \rightarrow f \bar{f}) \approx +\frac{4M^2g^2}{q^2 + m^2} \times \delta_{s',s_1} \delta_{s',s_2} + 0. \] (66)

In the non-relativistic normalization, these amplitudes become
\[ f(f f \rightarrow f f) \approx +\frac{Mg^2}{4\pi(q^2 + m^2)} \times \delta_{s',s_1} \delta_{s',s_2} - \frac{Mg^2}{4\pi(q^2 + m^2)} \times \delta_{s',s_1} \delta_{s',s_2}, \]
\[ f(f \bar{f} \rightarrow f \bar{f}) \approx +\frac{Mg^2}{4\pi(q^2 + m^2)} \times \delta_{s',s_1} \delta_{s',s_2}, \] (67)

which agree with Born amplitudes for
\[ \tilde{V}_{ff}(q) = \tilde{V}_{\bar{f}f} = -\frac{g^2}{q^2 + m^2}. \] (68)

Fourier transforming this formula back too coordinate space gives us the Yukawa potential
\[ V_{ff}(x_1 - x_2) = V_{f \bar{f}}(x_1 - x_2) = -\frac{g^2}{4\pi} \times \frac{e^{-mr}}{r} \quad \text{for} \quad r = |x_1 - x_2|. \] (69)

Note the signs of these potentials — two fermions attract each other, and a fermion and an antifermion also attract each other!