1. Consider the Wess–Zumino model comprising a Weyl spinor field $\chi^\alpha(x)$, a complex scalar field $\phi(x)$ and the Lagrangian

$$\mathcal{L} = i\bar{\chi}_\alpha \partial^\alpha \chi_\alpha + \partial^m \phi \partial_m \phi - \frac{1}{4} g \phi \chi^\alpha \chi_\alpha - g \frac{1}{2} \phi^* \bar{\chi}_\alpha \chi^\alpha - \frac{1}{4} \lambda |\phi|^4 + \text{counterterms.}$$

(a) Write down the Feynman rules for this theory.

(b) Write down all the counterterms and calculate their infinite parts at the one-loop order.

(c) Calculate the (one-loop) $\beta$–functions $\beta_g$ and $\beta_\lambda$ and show that $\lambda = g^2$ is a fixed line of the renormalization group flow.

Actually, the renormalization group preserves the $\lambda = g^2$ condition to all orders of perturbation theory because for $\lambda = g^2$ the Wess–Zumino model is supersymmetric.

2. Consider the so-called topologically massive Yang–Mills theory in three spacetime dimensions (two space, one time). The Lagrangian of this $d = 3$ theory

$$\mathcal{L} = -\frac{1}{2g^2} \text{tr}(F^{\mu\nu}F_{\mu\nu}) + \frac{n}{8\pi} \epsilon^{\lambda\mu\nu} \text{tr}(A_\lambda F_{\mu\nu} + \frac{2i}{3} A_\lambda A_\mu A_\nu)$$

(1)

is not quite gauge invariant, but it leads to a gauge invariant action.

(a) Show that the action $S = \int \mathcal{L} d^3x$ is invariant under infinitesimal gauge transformations of the vector field $A_\mu$.

The finite gauge transformation of this theory are beyond the scope of this exam.*

(b) Derive the classical field equation for the gauge field $A_\mu(x)$ and write it down in a gauge-covariant form.

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* FYI, under topologically non-trivial finite gauge transforms $U(x)$, even the action $S = \int \mathcal{L} d^3x$ is only invariant up to a constant. However, for integer coefficients $n$ of the second term in (1), $e^{iS}$ is gauge invariant, which is sufficient for the gauge invariance of the Functional Integral of the quantum theory.
(c) Focus on the linear terms in the field equation and show that $A_\mu(x)$ is a massive vector field of mass

$$M = \frac{ng^2}{4\pi}. \quad (2)$$

(In $d = 3$, $g$ has the dimension of $\sqrt{\text{mass}}$.)

(d) Write down the Feynman rules for the topologically massive Yang–Mills theory.

3. Finally, consider the fermionic QCD in three spacetime dimensions,

$$\mathcal{L} = \frac{-1}{2g^2} \text{tr}(F_{\mu\nu}F_{\mu\nu}) + \bar{\Psi}(i\slashed{D} - m)\Psi. \quad (3)$$

Note that in odd spacetime dimensions, massive Dirac fermions have no Parity symmetry. Consequently, quark loop diagrams yield parity-violating amplitudes for the gluons thanks to $\text{tr}(\gamma^\lambda\gamma^\mu\gamma^\nu) = 2i\epsilon^{\lambda\mu\nu}$ in $d = 3$ (and similar formulae in higher odd dimensions).

(a) Evaluate the one loop diagram

and show that for small gluon momentum $|k^2| \ll m^2$ it yields

$$\Sigma_{\psi\text{loop}}^{\mu\nu}(k) = \frac{g^2N_F}{8\pi} \left(ik_{\lambda\epsilon}^{\lambda\mu\nu} + \frac{k^{\mu}k^{\nu} - g^{\mu\nu}k^2}{3m} + O\left(\frac{k^3}{m^2}\right)\right) \quad (4)$$

(b) Similarly, show that for three external gluons with small momenta (compared to the fermions mass $m$)

$$\quad = \frac{g^3N_F}{8\pi} f^{abc}\epsilon^{\lambda\mu\nu} + O\left(\frac{k}{m}\right). \quad (5)$$

(c) Show that for quark loops with four or more external gluons with small momenta, all the one-quark-loop amplitudes are suppressed by negative powers of the quark mass $m$.  

2
Now consider the Functional Integral for the $d = 3$ QCD. Let us integrate $\int D[\Psi(x)] D[\overline{\Psi}(x)]$ over the quark fields for fixed gauge fields $A_{\mu}^a(x)$ and ghost fields $c^a(x)$, $\overline{c}^a(x)$. The result of this integration is an effective quantum theory of the gauge and ghost fields with action

$$S[A_{\mu}^a + \text{ghosts}] = S_M[A_{\mu}^a + \text{ghosts}] - i \log \Det(i \slashed{D} - m)$$

(d) Use the results of questions (a), (b) and (c) to show that in the large quark mass $m$ limit,

$$-i \log \Det(i \slashed{D} - m) = \int d^3 x \left\{ \frac{N_F}{8\pi} \epsilon^{\lambda\mu\nu} \text{tr} (A_\lambda \mathcal{F}_{\mu\nu} + \frac{2i}{3} A_\lambda A_{\mu} A_{\nu}) + O \left( \frac{1}{m} \right) \right\}$$

and consequently, the effective low-energy quantum theory is precisely the topologically massive Yang–Mills theory with $n = N_F$.

(e) Finally, show that changing the sign of the fermionic mass term (i.e., taking $m < 0$) results in a topologically massive Yang–Mills theory with $n = -N_F$. In a generic case of $N_F$ fermionic flavors with different masses of both signs,

$$n = \sum_{i=1}^{N_F} \text{sign}(m_i).$$