1. Consider coherent states of a harmonic oscillator.

(a) Show that for any complex number $\alpha$,

$$|\alpha\rangle \text{ def } = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) \, |0\rangle = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} |0\rangle \quad \text{and} \quad \hat{a} |\alpha\rangle = \alpha |\alpha\rangle. \quad (1)$$

(b) Calculate the uncertainties $\Delta q$ and $\Delta p$ for a coherent state $|\alpha\rangle$ and verify their minimality: $\Delta q \Delta p = \frac{1}{2} \hbar$. Also, verify $\delta n = \sqrt{n}$ where $n \text{ def } = \langle \hat{n} \rangle = |\alpha|^2$. 

Hint: use $\hat{a} |\alpha\rangle = \alpha |\alpha\rangle$ and $\langle \alpha | \hat{a}^\dagger = \alpha^* \langle \alpha |$.

(c) Show that for any initial coherent state $|\alpha_0\rangle$,

$$|\psi(t)\rangle \equiv e^{-i\omega t/2} |\alpha_0 e^{-i\omega t}\rangle \quad (2)$$

satisfies the time-dependent Schrödinger equation.

(d) The coherent states are not quite orthogonal to each other. Calculate their overlap.

Now consider coherent states of multi-oscillator systems and hence quantum fields. In particular, let us focus on creation and annihilation fields $\hat{\Psi}^\dagger(x)$ and $\hat{\Psi}(x)$ for non-relativistic spinless bosons.

(e) Generalize (a) and construct coherent states $|\Phi\rangle$ which satisfy

$$\hat{\Psi}(x) |\Phi\rangle = \Phi(x) |\Phi\rangle \quad (3)$$

for any given classical complex field $\Phi(x)$.

(f) Show that for any such coherent state, $\Delta N = \sqrt{N}$ where

$$\bar{N} \text{ def } = \langle \Phi | \hat{N} | \Phi \rangle = \int dx |\Phi(x)|^2. \quad (4)$$

(g) Let

$$\hat{H} = \int dx \left( \frac{\hbar^2}{2M} \nabla \hat{\Psi}^\dagger \cdot \nabla \hat{\Psi} + v(x) \hat{\Psi}^\dagger \hat{\Psi} \right)$$

and show that for any classical field configuration $\Phi(x, t)$ that satisfies the classical
field equation
\[ i\hbar \frac{\partial}{\partial t} \Phi(x, t) = \left( -\frac{\hbar^2}{2M} \nabla^2 + V(x) \right) \Phi(x, t), \]
the time-dependent coherent state \(|\Phi\rangle\) satisfies the true Schrödinger equation
\[ i\hbar \frac{\partial}{\partial t} |\Phi\rangle = \hat{H} |\Phi\rangle. \] (5)

(h) Finally, show that the quantum overlap \(|\langle \Phi_1 | \Phi_2 \rangle|^2\) between two different coherent states is exponentially small for any macroscopic difference \(\delta \Phi(x) = \Phi_1(x) - \Phi_2(x)\) between the two field configurations.

2. Consider a complex relativistic field \(\Phi(x)\) with a Lagrangian density
\[ L = \partial^\mu \Phi^* \partial_\mu \Phi - m^2 \Phi^* \Phi - \frac{1}{4} \lambda (\Phi^* \Phi)^2. \] (6)
This Lagrangian has a symmetry \(\Phi(x) \mapsto e^{i\theta} \Phi(x)\). According to Noether theorem (which we shall study later in class), this symmetry gives rise to a conserved current
\[ J^\mu = i \Phi^* \partial^\mu \Phi - i (\partial^\mu \Phi^*) \Phi. \] (7)

(a) Write down classical field equations for \(\Phi(x)\) and \(\Phi^*(x)\) (treat them as independent fields!) and verify that indeed \(\partial_\mu J^\mu = 0\).

Canonical quantization of the complex field yields non-hermitian quantum fields \(\hat{\Phi}(x) \neq \hat{\Phi}^\dagger(x)\) and \(\hat{\Pi}(x) \neq \hat{\Pi}^\dagger(x)\) and the Hamiltonian
\[ \hat{H} = \int d^3x \left( \hat{\Pi}^\dagger \hat{\Pi} + \nabla \hat{\Phi}^\dagger \cdot \nabla \hat{\Phi} + m^2 \hat{\Phi}^\dagger \hat{\Phi} + \frac{1}{4} \lambda \hat{\Phi}^\dagger \hat{\Phi}^\dagger \hat{\Phi} \hat{\Phi} \right). \] (8)

(b) Derive the Hamiltonian (8) and write down the equal-time commutation relations between the quantum fields \(\hat{\Phi}(x), \hat{\Phi}^\dagger(x), \hat{\Pi}(x)\) and \(\hat{\Pi}^\dagger(x)\).
Because of the non-hermiticity of the quantum fields $\hat{\Phi}(x) \neq \hat{\Phi}^\dagger(x)$ and $\hat{\Pi}(x) \neq \hat{\Pi}^\dagger(x)$, their respective plane-wave modes $\hat{\Phi}_p, \hat{\Phi}^\dagger_p, \hat{\Pi}_p$ and $\hat{\Pi}^\dagger_p$ are completely independent of each other i.e., $\hat{\Phi}^\dagger_p \neq \hat{\Phi}_{-p}$ and $\hat{\Pi}^\dagger_p \neq \hat{\Pi}_{-p}$. Let us therefore define:

\begin{align*}
\hat{a}_p & \equiv \frac{E_p\hat{\Phi}_p + i\hat{\Pi}^\dagger_p}{\sqrt{2E_p}}, \\
\hat{a}^\dagger_p & \equiv \frac{E_p\hat{\Phi}^\dagger_p - i\hat{\Pi}_p}{\sqrt{2E_p}}, \\
\hat{b}_p & \equiv \frac{E_p\hat{\Phi}^\dagger_p - i\hat{\Pi}_p}{\sqrt{2E_p}}, \\
\hat{b}^\dagger_p & \equiv \frac{E_p\hat{\Phi}_p + i\hat{\Pi}^\dagger_p}{\sqrt{2E_p}},
\end{align*}

where

$$E_p \equiv \sqrt{p^2 + m^2}. \tag{10}$$

(c) Verify the bosonic commutation relations (at equal times) between the annihilation operators $\hat{a}_p$ and $\hat{b}_p$ and the corresponding creation operators $\hat{a}^\dagger_p$ and $\hat{b}^\dagger_p$.

(d) Now, let us turn off the interactions (i.e., set $\lambda = 0$). Show that the Hamiltonian of free charged fields is

\begin{align*}
\hat{H}_{\text{free}} & \equiv \int d^3x \left( \hat{\Pi}^\dagger \hat{\Pi} + \nabla \hat{\Phi}^\dagger \cdot \nabla \hat{\Phi} + m^2 \hat{\Phi}^\dagger \hat{\Phi} \right) \\
& = \int \frac{d^3p}{(2\pi)^3} E_p \left( \hat{a}^\dagger_p \hat{a}_p + \hat{b}^\dagger_p \hat{b}_p \right) + \text{const}. \tag{11}
\end{align*}

(e) Next, consider the electric charge operator $\hat{Q} = \int d^3x \: \hat{J}_0(x)$. Show that for the system at hand

$$\hat{Q} = \int d^3x \left( \frac{i}{2} \{ \hat{\Pi}^\dagger, \hat{\Phi}^\dagger \} - \frac{i}{2} \{ \hat{\Pi}, \hat{\Phi} \} \right) = \int \frac{d^3p}{(2\pi)^3} \left( \hat{a}^\dagger_p \hat{a}_p - \hat{b}^\dagger_p \hat{b}_p \right). \tag{12}$$

Actually, the classical formula (7) for the current $J_\mu(x)$ determines eq. (12) only up to ordering of the non-commuting operators $\hat{\Pi}(x)$ and $\hat{\Phi}(x)$ (and likewise of the $\hat{\Pi}^\dagger(x)$ and $\hat{\Phi}^\dagger(x)$). The anti-commutators in eq. (12) provide a solution to this ordering ambiguity, but any other ordering would be just as legitimate.
The net effect of changing operator ordering in $\hat{J}_0$ amounts to changing the total charge $\hat{Q}$ by an infinite constant (prove this!). The specific ordering in eq. (12) provides for the neutrality of the vacuum state.

Finally, consider the stress-energy tensor for the complex field $\Phi(x)$. Classically, Noether theorem gives

$$T^{\mu\nu} = \partial^\mu \Phi^* \partial^\nu \Phi + \partial^\mu \Phi \partial^\nu \Phi^* - g^{\mu\nu} \mathcal{L}.$$  

(13)

Quantization of this formula is straightforward (modulo ordering ambiguity); for example, $\hat{\mathcal{H}} \equiv \hat{T}^{00}$ is precisely the integrand on the right hand side of eq. (8).

(f) Consider the total mechanical momentum operator of the fields $\hat{P}_i^{\text{mech}} = \int d^3x \hat{T}^{0i}(x)$ and show that in terms of creation and annihilation operators

$$\hat{\mathbf{P}}_{\text{mech}} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \mathbf{p} \left( \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} + \hat{b}^\dagger_{\mathbf{p}} \hat{b}_{\mathbf{p}} \right)$$  

(14)

Physically, eqs. (14), (11) and (12) show that a complex field $\Phi(x)$ describes both a particle and its antiparticle; they have exactly the same rest mass $m$ but exactly opposite charges $\pm 1$. 

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