1. Consider the matrix \( \gamma^5 \overset{\text{def}}{=} i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \).

(a) Show that \( \gamma^5 \) anticommutes with each of the \( \gamma^\mu \) matrices, \( \gamma^5 \gamma^\mu = -\gamma^\mu \gamma^5 \).
(b) Show that \( \gamma^5 \) is hermitian and that \( (\gamma^5)^2 = 1 \).
(c) Show that \( \gamma^5 = (\frac{-i}{24}) \epsilon^{\kappa\lambda\mu\nu} \gamma_\kappa \gamma_\lambda \gamma_\mu \gamma_\nu \) and \( \gamma^{[\kappa_\gamma^\lambda \gamma^\mu \gamma^\nu]} = -i\epsilon^{\kappa\lambda\mu\nu} \gamma^5 \).

(Sign convention: \( \epsilon^{0123} = +1, \epsilon^{0123} = -1 \).)
(d) Show that \( \gamma^{[\lambda_\gamma^\mu \gamma^\nu]} = i\epsilon^{\kappa\lambda\mu\nu} \gamma_\kappa \gamma^5 \).
(e) Show that any 4 \times 4 matrix \( \Gamma \) is a unique linear combination of the following 16 matrices: 1, \( \gamma^\mu \), \( \gamma^{[\mu_\gamma^\nu]} \), \( \gamma^5 \gamma^\mu \) and \( \gamma^5 \).

Under continuous Lorentz symmetries, Dirac spinor fields \( \Psi(x) \) transform according to \( \Psi'(x') = M(L)\Psi(x = L^{-1}x') \) where \( M(L = e^{\theta}) = \exp(-\frac{i}{2}\theta_{\alpha\beta}S^{\alpha\beta}) \). Consider the transformation rules for the independent bilinears \( \bar{\Psi} \Gamma \Psi \), namely (cf. (e))

\[
S = \bar{\Psi}\Psi, \quad V^\mu = \bar{\Psi}\gamma^\mu\Psi, \quad T^{\mu\nu} = \bar{\Psi}\gamma^{[\mu_\gamma^\nu]}\Psi, \quad A^\mu = \bar{\Psi}\gamma^5\gamma^\mu\Psi \quad \text{and} \quad P = \bar{\Psi}\gamma^5\Psi.
\]

(f) Show that under continuous Lorentz symmetries, the \( S \) and the \( P \) transform as scalars, the \( V^\mu \) and the \( A^\mu \) as vectors and the \( T^{\mu\nu} \) as an antisymmetric tensor.

2. Under the parity symmetry \( \mathcal{P} : (x, t) \mapsto (-x, t) \), Dirac spinor fields transform according to

\[
\mathcal{P}\Psi(x, t)\mathcal{P} \equiv \Psi'(x, t) = \pm \gamma^0 \Psi(-x, t)
\]

where the overall sign depends on the so-called intrinsic parity of a particular Dirac field. Note: \( \mathcal{P} \) here is a unitary operator in the fermionic Fock space; by nature of the parity symmetry, \( \mathcal{P}^2 = 1 \).

(a) Verify the covariance of the Dirac equation under this symmetry.
(b) Find the transformation rules of the bilinears (1) under parity and show that while \( S \) is a true scalar and \( V \) is a true (polar) vector, \( P \) is a pseudoscalar and \( A \) is an axial vector.
3. In theories involving both bosons and fermions, one often has to combine commutation and anti-commutation relations of various operators, depending on the overall statistics of the operators involved. For that purpose, it is useful to define a ‘mixed’ commutator bracket

\[ [\hat{A}, \hat{B}] \overset{\text{def}}{=} \hat{A}\hat{B} - (-1)^{AB} \hat{B}\hat{A} \]  

(3)

where \((-1)^{AB}\) is \(-1\) if both \(\hat{A}\) and \(\hat{B}\) have overall Fermi statistics (i.e., each comprises an odd number of fermionic creation/annihilation operators — the number of bosonic creation/annihilation operators does not matter) and \(+1\) in all other cases.

(a) Verify the Leibniz rule for the mixed brackets: 

\[ [\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}][\hat{C}] + (-1)^{AB}[\hat{B}, [\hat{A}, \hat{C}]] \]

and write down a similar rule for the \([\hat{A}\hat{B}, \hat{C}]\).

(b) Similarly, express \([\hat{A}\hat{B}, \hat{C}\hat{D}]\) in terms of appropriate mixed brackets of \(\hat{A}\) or \(\hat{B}\) with \(\hat{C}\) or \(\hat{D}\).

(c) Prove the ‘mixed’ Jacobi identity

\[ (-1)^{CA}[\hat{A}, [\hat{B}, \hat{C}]] + (-1)^{AB}[[\hat{B}, \hat{C}], \hat{A}] + (-1)^{BC}[[\hat{C}, \hat{A}], \hat{B}] = 0. \]  

(4)

In other words (and notations),

\[
\begin{align*}
[\hat{B}_1, [\hat{B}_2, \hat{B}_3]] &+ [\hat{B}_2, [\hat{B}_3, \hat{B}_1]] + [\hat{B}_3, [\hat{B}_1, \hat{B}_2]] = 0, \\
[\hat{B}_1, [\hat{B}_2, \hat{F}]] &+ [\hat{B}_2, [\hat{F}, \hat{B}_1]] + [\hat{F}, [\hat{B}_1, \hat{B}_2]] = 0, \\
\{\hat{F}_1, [\hat{F}_2, \hat{B}]\} &- \{\hat{F}_2, [\hat{B}, \hat{F}_1]\} + [\hat{B}, \{\hat{F}_1, \hat{F}_2\}] = 0, \\
[\hat{F}_1, \{\hat{F}_2, \hat{F}_3\}] &+ [\hat{F}_2, \{\hat{F}_3, \hat{F}_1\}] + [\hat{F}_3, \{\hat{F}_1, \hat{F}_2\}] = 0,
\end{align*}
\]  

(5)

where ‘\(B\)’ and ‘\(F\)’ indicate the overall statistics of the operator involved.

4. Finally, an exercise in fermionic creation and annihilation operators and their anticommutation relations,

\[ \{\hat{a}_{\alpha}, \hat{a}_{\beta}\} = \{\hat{a}_{\alpha}^\dagger, \hat{a}_{\beta}^\dagger\} = 0, \quad \{\hat{a}_{\alpha}, \hat{a}_{\beta}^\dagger\} = \delta_{\alpha,\beta}. \]  

(6)

(a) Calculate the commutators \([\hat{a}_{\alpha}^\dagger \hat{a}_{\beta}, \hat{a}_{\gamma}^\dagger], [\hat{a}_{\alpha}^\dagger \hat{a}_{\beta}, \hat{a}_{\delta}]\) and \([\hat{a}_{\alpha}^\dagger \hat{a}_{\beta}, \hat{a}_{\gamma}^\dagger \hat{a}_{\delta}]\).
(b) Consider two one-body operators $\hat{A}_1$ and $\hat{B}_1$ and let $\hat{C}_1$ be their commutator, $\hat{C}_1 = [\hat{A}_1, \hat{B}_1]$. Let $\hat{A}$ be the second-quantized forms of $\hat{A}_{\text{tot}}$,

$$\hat{A} = \sum_{\alpha, \beta} \langle \alpha | \hat{A}_1 | \beta \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta,$$

(7)

and ditto for the second-quantized $\hat{B}$ and $\hat{C}$.

Verify that $[\hat{A}, \hat{B}] = \hat{C}$.

(c) Calculate the commutator $[\hat{a}_\mu^\dagger \hat{a}_\nu, \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta]$.

(d) The second quantized form of a two-body additive operator

$$\hat{B}_{\text{tot}} = \frac{1}{2} \sum_{i \neq j} \hat{B}_2(i^{\text{th}} \text{ and } j^{\text{th}} \text{ particles})$$

acting on identical fermions is

$$\hat{B} = \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} \langle \alpha \otimes \beta | \hat{B}_2 | \gamma \otimes \delta \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\delta \hat{a}_\gamma.$$

(8)

This expression is similar to its bosonic counterpart, but note the reversed order of the annihilation operators $\hat{a}_\delta$ and $\hat{a}_\gamma$.

Consider a one-body operator $\hat{A}_1$ and two-body operator $\hat{B}_2$ and $\hat{C}_2$ where $\hat{C}_2 = \left[ (\hat{A}_1(1^{\text{st}}) + \hat{A}_1(2^{\text{nd}})), \hat{B}_2 \right]$. Show that the respective second-quantized operators in the fermionic Fock space satisfy $\hat{C} = [\hat{A}, \hat{B}]$. 

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