1. In the linear sigma model (cf. previous homework), calculate the tree-level decay rate \( \Gamma(\sigma \to \pi \pi) \).

2. Consider two Dirac fields \( \Psi_1(x) \) and \( \Psi_2(x) \) coupled to a real scalar field \( \Phi(x) \):

\[
\mathcal{L} = \bar{\Psi}_1(i \not\partial - m_1)\Psi_1 + \bar{\Psi}_2(i \not\partial - m_2)\Psi_2 + \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2}M^2\Phi^2 - g_1\bar{\Phi}\not\psi_1\psi_1 - g_2\bar{\Phi}\not\psi_2\psi_2.
\]

At the tree level, calculate the matrix element, the partial cross-section and the total cross-section for scattering of one type of a fermion off the other type, \( f_1 + f_2 \to f_1 + f_2 \). Take the initial fermions to be unpolarized and sum over the final fermion’s polarizations.

Hint: Prove and use

\[
\frac{1}{2} \sum_{s,s'} \left| \bar{u}(p',s')u(p,s) \right|^2 = \frac{1}{2} \text{tr} \left( (m + \not{p})(m + \not{p}) \right)
= 2(m^2 + EE' - pp').
\]

3. Consider Rutherford scattering of an electron off a static point-like electric charge. See problem 4.4 of the Peskin & Schroeder textbook for details and follow steps (a), (b) and (c) therein.

(d) In addition, calculate the scattering cross-section for a relativistic unpolarized electron.

Hint: Prove and use

\[
\frac{1}{2} \sum_{s,s'} \left| \bar{u}(p',s')\gamma^0u(p,s) \right|^2 = \frac{1}{2} \text{tr} \left( (m + \not{p})\gamma^0(m + \not{p})\gamma^0 \right)
= 2(m^2 + EE' + pp').
\]