Problem 1(a):
\[\gamma^\mu\gamma^\nu = \pm \gamma^\nu\gamma^\mu\] where the sign is ‘+’ for \(\mu = \nu\) and ‘−’ otherwise. Hence for any product \(\Gamma\) of the \(\gamma\) matrices, \(\gamma^\mu\Gamma = (-1)^{n_\mu}\gamma^\mu\) where \(n_\mu\) is the number of \(\gamma^\nu\neq\gamma^\mu\) factors of \(\Gamma\). For \(\Gamma = \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3\), \(n_\mu = 3\) for any \(\mu = 0, 1, 2, 3\); thus \(\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu\).

Problem 1(b):
First,
\begin{align*}
(\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3)^\dagger &= -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger = +i\gamma^3\gamma^2\gamma^1\gamma^0 \\
&= +i(\gamma^3\gamma^2\gamma^1\gamma^0) = (-1)^3i\gamma^0((\gamma^3\gamma^2)^\dagger) \\
&= (-1)^3+2i\gamma^0(\gamma^1(\gamma^3\gamma^2)) = (-1)^3+2+1i\gamma^0(\gamma^1(\gamma^2\gamma^3)) \\
&= +i\gamma^0\gamma^1\gamma^2\gamma^3 \equiv +\gamma^5.
\end{align*}

Second,
\begin{align*}
(\gamma^5)^2 &= \gamma^5(\gamma^5)^\dagger = (i\gamma^0\gamma^1\gamma^2\gamma^3)(i\gamma^3\gamma^2\gamma^1\gamma^0) = -\gamma^0\gamma^1\gamma^2(\gamma^3\gamma^2)\gamma^2\gamma^1\gamma^0 \\
&= +\gamma^0\gamma^1(\gamma^2\gamma^2)\gamma^1\gamma^0 = -\gamma^0(\gamma^1\gamma^1)\gamma^0 = +\gamma^0\gamma^0 = +1.
\end{align*}

Problem 1(c):
Any four distinct \(\gamma^\kappa, \gamma^\lambda, \gamma^\mu,\) and \(\gamma^\nu\) are \(\gamma^0, \gamma^1, \gamma^2,\) and \(\gamma^3\) in some order. They all anticommute with each other, hence \(\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu = \epsilon^{\kappa\lambda\mu\nu}\gamma^0\gamma^1\gamma^2\gamma^3 \equiv -i\epsilon^{\kappa\lambda\mu\nu}\gamma^5\). The rest is obvious.

Problem 1(d):
\begin{align*}
ie^{\kappa\lambda\mu\nu}\gamma_\kappa\gamma^5 &= \gamma_\kappa\gamma^{[\kappa\lambda\mu\nu]} \\
&= \frac{1}{4}\gamma_\kappa \left(\gamma^{[\kappa\mu\nu]}\gamma^\lambda - \gamma^{[\lambda\mu]}\gamma^{[\nu\kappa]} + \gamma^{[\lambda\nu\mu]}\gamma^\kappa - \gamma^{[\kappa\mu\nu]}\gamma^\lambda\right) \\
&= \frac{1}{4}(4\gamma^{[\lambda\mu\nu]} + 2\gamma^{[\lambda\mu\nu]} + 4\gamma^{[\lambda\mu\nu]} + 2\gamma^{[\nu\mu\lambda]}) \\
&= \frac{1}{4}(4 + 2 + 0 - 2)\gamma^{[\lambda\mu\nu]} = \gamma^{[\lambda\mu\nu]}.
\end{align*}
Problem 1(e):

**Proof by inspection:** In the Weyl basis, the 16 matrices are

\[
1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^\mu = \begin{pmatrix} \sigma^\mu \\ 0 \end{pmatrix}, \quad \gamma^{[\mu, \nu]} = \begin{pmatrix} 0 & \sigma^{[\nu} \\ \sigma^{\mu]} & 0 \end{pmatrix}, \quad \gamma^{5, \gamma^\mu} = \begin{pmatrix} 0 & -\sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix},
\]

and their linear independence is self-evident. Since there are only 16 independent $4 \times 4$ matrices altogether, any such matrix $\Gamma$ is a linear combination of the matrices (S.4). \textit{Q.E.D.}

**Algebraic Proof:** Without making any assumption about the matrix form of the $\gamma^\mu$ operators, let us consider the Clifford algebra $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$. Because of these anticommutation relations, one may re-order any product of the $\gamma$’s as $\pm (\gamma^0 \text{ or } 1) \times (\gamma^1 \text{ or } 1) \times (\gamma^2 \text{ or } 1) \times (\gamma^3 \text{ or } 1)$. The net result is (up to a sign or $\pm i$ factor) one of the 16 operators $1$, $\gamma^\mu$, $\gamma^{[\mu, \nu]}$, $\gamma^{5, \gamma^\mu}$ (cf. (d)) or $\gamma^5$ (cf. (c)). Consequently, any operator $\Gamma$ algebraically constructed of the $\gamma^\mu$’s is a linear combination of these 16 operators.

Incidentally, the algebraic argument explains why the $\gamma^4$ (and hence all their products) should be realized as $4 \times 4$ matrices since any lesser matrix size would not accommodate 16 independent products. That is, the $\gamma$’s are $4 \times 4$ matrices in four spacetime dimensions; different dimensions call for different matrix sizes. Specifically, in spacetimes of even dimensions $d$, there are $2^d$ independent products of the $\gamma$ operators, so we need matrices of size $2^{d/2} \times 2^{d/2}$: $2 \times 2$ in two dimensions, $4 \times 4$ in four, $8 \times 8$ in six, $16 \times 16$ in eight, $32 \times 32$ in ten, etc., etc.

In odd dimensions, there are only $2^{d-1}$ independent operators because $\gamma^{d+1} \equiv (i)\gamma^0 \gamma^1 \cdots \gamma^{d-1}$ — the analogue of the $\gamma^5$ operator in $4d$ — commutes rather than anticommutes with all the $\gamma^\mu$ and hence with the whole algebra. Consequently, one has two distinct representations of the Clifford algebra — one with $\gamma^{d+1} = +1$ and one with $\gamma^{d+1} = -1$ — but in each representation there are only $2^{d-1}$ independent operator products, which call for the matrix size of $2^{(d-1)/2} \times 2^{(d-1)/2}$. For example, in three spacetime dimensions (two space, one time), can take $(\gamma^0, \gamma^1, \gamma^2) = (\sigma_3, i\sigma_1, i\sigma_2)$ for $\gamma^4 \equiv i\gamma^0 \gamma^1 \gamma^2 = +1$ or $(\gamma^0, \gamma^1, \gamma^2) = (\sigma_3, i\sigma_1, -i\sigma_2)$ for $\gamma^4 = -1$, but in both cases we have $2 \times 2$ matrices.

Problem 1(f):

Under a continuous Lorentz symmetry $x \mapsto x' = Lx$, the Dirac spinor field and its conjugate
transfrom according to
\[ \Psi'(x') = M(L)\Psi(x = L^{-1}x'), \quad \overline{\Psi}(x') = \overline{\Psi}(x = L^{-1}x')M^{-1}(L), \quad (S.5) \]
hence any bilinear \( \overline{\Psi}\Gamma\Psi \) transforms according to
\[ \overline{\Psi}(x')\Gamma\Psi(x') = \overline{\Psi}(x)\Gamma'\Psi(x) \quad (S.6) \]
where
\[ \Gamma' = M^{-1}(L)\Gamma M(L). \quad (S.7) \]
Obviously, for \( \Gamma = 1 \), \( \Gamma' = M^{-1}M = 1 \). According to previous homeworks, for \( \Gamma = \gamma^{\mu} \), \( \Gamma' = M^{-1}\gamma^{\mu}M = L^{\mu}_{\nu}\gamma^{\nu} \). Similarly, \( M^{-1}\gamma^{\mu}\gamma^{\nu}M = L^{\mu}_{\alpha}\gamma^{\nu}\gamma^{\beta} \) and hence for \( \Gamma = \gamma^{[\mu}\gamma^{\nu]} \), \( \Gamma' = L^{\mu}_{\alpha}L^{\nu}_{\beta}\gamma^{[\alpha}\gamma^{\beta]} \). Consequently,
\[ S'(x') = S(x), \quad V^{\mu}(x') = L^{\mu}_{\nu}V^{\nu}(x), \quad T^{\mu\nu}(x') = L^{\mu}_{\alpha}L^{\nu}_{\beta}T^{\alpha\beta}(x), \quad (S.8) \]
which makes \( S \) a Lorentz scalar, \( V^{\mu} \) a Lorentz vector and \( T^{\mu\nu} \) a Lorentz tensor (with two antisymmetric indices).

The \( \gamma^5 \) matrix commutes with even products of the \( \gamma^{\mu} \) matrices such as \( \gamma^{\mu}\gamma^{\nu} \), hence it commutes with all \( S^{\mu\nu} \) and therefore with \( M(L) = \exp\left(-\frac{i}{2}\theta_{\mu\nu}S^{\mu\nu}\right) \). Consequently, for \( \Gamma = \gamma^5 \), \( \Gamma' = M^{-1}\gamma^5M = \gamma^5 \) while for \( \Gamma = \gamma^5\gamma^{\mu} \), \( \Gamma' = M^{-1}\gamma^5\gamma^{\mu}M = \gamma^5M^{-1}\gamma^{\mu}M = \gamma^5(L^{\mu}_{\nu}\gamma^{\nu}) = L^{\mu}_{\nu}(\gamma^5\gamma^{\nu}) \). Therefore,
\[ P'(x') = P(x), \quad A^{\mu}(x') = L^{\mu}_{\nu}A^{\nu}(x), \quad (S.9) \]
which makes \( P \) a Lorentz scalar and \( A \) a Lorentz vector.  

**Problem 2(a):**
Given \( \Psi'(x',t) = \pm\gamma^{0}\Psi(x = -x',t' = t) \), we have
\[
(i\,\not\!p' - m)\Psi'(x') \equiv (i\gamma^{0}\partial_0 + i\vec{\gamma} \cdot \nabla' - m)(\pm\gamma^{0})\Psi(x',t) = (\pm\gamma^{0})(i\gamma^{0}\partial_0 - i\vec{\gamma} \cdot \nabla - m)\Psi(x',t) \\
= (\pm\gamma^{0})(i\gamma^{0}\partial_0 + i\vec{\gamma} \cdot \nabla - m)\Psi(-x,t) \\
\equiv (\pm\gamma^{0})(i\,\not\!p - m)\Psi\bigg|_{x'}, \quad (S.10)
\]
Problem 2(b):

Parity properties of the Dirac bilinears (1) follow from the commutation relations of the 16 operators (1e) with the $\gamma^0$. It is easy to verify that the $1, \gamma^0, \gamma^i\gamma^j$ and $\gamma^5\gamma^i$ commute with the $\gamma^0$ while the $\gamma^i, \gamma^0\gamma^i, \gamma^5\gamma^0$ and $\gamma^5$ anticommute with the $\gamma^0$. Consequently,

- the $S, V^0, T^i j$ and $A^i$ remain invariant under parity, while
- the $V_i, T^0i, A^0$ and $P$ change their signs.

In three-dimensional terms, this means that $S$ and $V^0$ are true scalars, $P$ and $A^0$ are pseudoscalars, $V$ is a true or polar vector, $A$ is a pseudovector or axial vector, and the tensor $T$ contains one true vector $T^0i$ and one axial vector $\frac{1}{2}\epsilon^{ijk}T_{jk}$. In space-time terms, we call $S$ a (Lorentz) (true) scalar, $P$ a (Lorentz) pseudoscalar, $V^\mu$ a (Lorentz) (true) vector and $A^\mu$ an (Lorentz) axial vector. Pedantically speaking, $T^{\mu\nu}$ is a Lorentz true tensor while $\tilde{T}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}T_{\alpha\beta}$ is a Lorentz pseudotensor, but few people are that pedantic.

Problem 3(a):

The overall statistics of the operator product $\hat{B}\hat{C}$ corresponds to $(-1)^{A(BC)} = (-1)^{AB}(-1)^{AC}$. Therefore,

$$[\hat{A}, \hat{B}\hat{C}] \defeq \hat{A}\hat{B}\hat{C} - (-1)^{AB}(-1)^{AC}\hat{B}\hat{C}\hat{A}$$

$$= \left(\hat{A}\hat{B} - (-1)^{AB}\hat{B}\hat{A}\right)\hat{C} + (-1)^{AB}\hat{B}\left(\hat{A}\hat{C} - (-1)^{AC}\hat{C}\hat{A}\right)$$

$$= [\hat{A}, \hat{B}]\hat{C} + (-1)^{AB}\hat{B}[\hat{A}, \hat{C}] \tag{S.11}$$

Likewise,

$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}\hat{B}\hat{C} - (-1)^{AC}(-1)^{BC}\hat{C}\hat{A}\hat{B}$$

$$= \hat{A}\left(\hat{B}\hat{C} - (-1)^{BC}\hat{C}\hat{B}\right) + (-1)^{BC}\left(\hat{A}\hat{C} - (-1)^{AC}\hat{C}\hat{A}\right)\hat{B}$$

$$= \hat{A}[\hat{B}, \hat{C}] + (-1)^{BC}[\hat{A}, \hat{C}]\hat{B} \tag{S.12}$$

Problem 3(b):

$$[\hat{A}\hat{B}, \hat{C}\hat{D}] = \hat{A}[\hat{B}, \hat{C}\hat{D}] + (-1)^{BC}(-1)^{BD}[\hat{A}, \hat{C}\hat{D}]\hat{B}$$

$$= \hat{A}[\hat{B}, \hat{C}]\hat{D} + (-1)^{BC}\hat{A}\hat{C}[\hat{B}, \hat{D}]$$

$$+ (-1)^{BC}(-1)^{BD}[\hat{A}, \hat{C}]\hat{D}\hat{B} + (-1)^{AC}(-1)^{BC}(-1)^{BD}\hat{C}[\hat{A}, \hat{D}]\hat{B} \tag{S.13}$$
Problem 3(c):

\[
(-1)^C A \hat{A} \hat{B} \hat{C} = (-1)^B C \hat{B} \hat{C} \hat{A} - (-1)^A B \hat{A} \hat{C} \hat{B},
\]

\[
(-1)^B C \hat{A} \hat{C} \hat{B} = (-1)^A B \hat{C} \hat{A} \hat{B} - (-1)^C A \hat{B} \hat{A} \hat{C},
\]

\[
(-1)^A B \hat{B} \hat{A} \hat{C} = (-1)^C A \hat{A} \hat{B} \hat{C} - (-1)^B C \hat{A} \hat{C} \hat{B}.
\]

Upon adding these three equations together, their right hand sides cancel out while the left hand sides add up to the Jacobi identity (4).

Problem 4(a):
Using the Leibnitz rules (S.11) and (S.12) and the anticommutation relations (6), the calculation is straightforward.

\[
[\hat{a}^\dagger_{\alpha} \hat{a}_{\beta}, \hat{a}^\dagger_{\gamma} \hat{a}_{\delta}] = \delta_{\beta\gamma} \hat{a}^\dagger_{\alpha} \hat{a}_{\delta},
\]

\[
[\hat{a}^\dagger_{\alpha} \hat{a}_{\beta}, \hat{a}^\dagger_{\delta} \hat{a}_{\delta}] = -\delta_{\alpha\delta} \hat{a}^\dagger_{\alpha} \hat{a}_{\beta},
\]

\[
[\hat{a}^\dagger_{\alpha} \hat{a}_{\beta}, \hat{a}^\dagger_{\delta} \hat{a}_{\gamma}] = \delta_{\beta\gamma} \hat{a}^\dagger_{\alpha} \hat{a}_{\delta} - \delta_{\alpha\delta} \hat{a}^\dagger_{\beta} \hat{a}_{\gamma}.
\]

Problem 4(b):
According to eq. (S.15), the commutator \([\hat{a}^\dagger_{\alpha} \hat{a}_{\beta}, \hat{a}^\dagger_{\gamma} \hat{a}_{\delta}]\) has exactly the same form as its bosonic counterpart. Hence, the proof of \([\hat{A}, \hat{B}] = \hat{C}\) proceeds exactly as in the bosonic case, cf. homework set #1 (problem 3(b)).

Problem 4(c):
Using the Leibnitz rules and eqs. (S.15),

\[
[\hat{a}^\dagger_{\mu} \hat{a}_{\nu}, \hat{a}^\dagger_{\alpha} \hat{a}^\dagger_{\beta} \hat{a}_{\gamma} \hat{a}_{\delta}] = \delta_{\nu\alpha} \hat{a}^\dagger_{\beta} \hat{a}^\dagger_{\gamma} \hat{a}_{\delta} + \delta_{\nu\beta} \hat{a}^\dagger_{\mu} \hat{a}^\dagger_{\alpha} \hat{a}_{\gamma} \hat{a}_{\delta} - \delta_{\mu\gamma} \hat{a}^\dagger_{\alpha} \hat{a}^\dagger_{\beta} \hat{a}_{\nu} \hat{a}_{\delta} - \delta_{\mu\delta} \hat{a}^\dagger_{\alpha} \hat{a}^\dagger_{\beta} \hat{a}_{\gamma} \hat{a}_{\nu}.
\]

Problem 4(d):
Again, we have a fermionic analogue to the bosonic second-quantized operators we studied in
homework set#1 (problem 3(d)). Given eqs. (7) and (S.16) (in which we exchange \( \gamma \leftrightarrow \delta \)), we have

\[
[\hat{A}, \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\delta^\dagger \hat{a}_\gamma] = \sum_{\mu, \nu} \langle \mu | \hat{A}_1 | \nu \rangle \left[ \hat{a}_\mu^\dagger \hat{a}_\nu, \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\delta^\dagger \hat{a}_\gamma \right]
\]

\[
= \sum_{\mu} \langle \mu | \hat{A}_1 | \alpha \rangle \hat{a}_\mu^\dagger \hat{a}_\beta^\dagger \hat{a}_\delta^\dagger \hat{a}_\gamma + \sum_{\mu} \langle \mu | \hat{A}_1 | \beta \rangle \hat{a}_\alpha^\dagger \hat{a}_\mu^\dagger \hat{a}_\delta^\dagger \hat{a}_\gamma
\]

\[
- \sum_{\nu} \langle \delta | \hat{A}_1 | \nu \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\nu \hat{a}_\gamma - \sum_{\nu} \langle \gamma | \hat{A}_1 | \nu \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\nu \hat{a}_\gamma
\]

and consequently, in light of eq. (8),

\[
[\hat{A}, \hat{B}] = \sum_{\alpha, \beta, \gamma, \delta} \langle \alpha \otimes \beta | \hat{B}_2 | \gamma \otimes \delta \rangle \left[ \sum_{\mu} \langle \mu | \hat{A}_1 | \alpha \rangle \hat{a}_\mu^\dagger \hat{a}_\beta^\dagger \hat{a}_\delta^\dagger \hat{a}_\gamma + \sum_{\mu} \langle \mu | \hat{A}_1 | \beta \rangle \hat{a}_\alpha^\dagger \hat{a}_\mu^\dagger \hat{a}_\delta^\dagger \hat{a}_\gamma \right.
\]

\[
- \sum_{\nu} \langle \delta | \hat{A}_1 | \nu \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\nu \hat{a}_\gamma - \sum_{\nu} \langle \gamma | \hat{A}_1 | \nu \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\nu \hat{a}_\gamma
\]

\[
= \sum_{\mu, \beta, \gamma, \delta} \langle \mu | \otimes | \hat{A}_1(1^{st}) \hat{B}_2 | \gamma \otimes \delta \rangle \hat{a}_\mu^\dagger \hat{a}_\beta^\dagger \hat{a}_\delta^\dagger \hat{a}_\gamma
\]

\[
+ \sum_{\alpha, \mu, \gamma, \delta} \langle \alpha | \otimes | \hat{A}_1(2^{nd}) \hat{B}_2 | \gamma \otimes \delta \rangle \hat{a}_\alpha^\dagger \hat{a}_\mu^\dagger \hat{a}_\delta^\dagger \hat{a}_\gamma
\]

\[
- \sum_{\alpha, \beta, \gamma, \nu} \langle \alpha | \otimes | \hat{B}_2 \hat{A}_1(2^{nd}) | \gamma \otimes \nu \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\nu \hat{a}_\gamma
\]

\[
- \sum_{\alpha, \beta, \nu, \delta} \langle \alpha | \otimes | \hat{B}_2 \hat{A}_1(1^{st}) | \nu \otimes \delta \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\nu \hat{a}_\gamma
\]

\langle\langle\text{renaming indices}\rangle\rangle

\[
= \sum_{\alpha, \beta, \gamma, \delta} \langle \alpha | \otimes | \hat{B}_2 | \gamma \otimes \delta \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \times \hat{a}_\delta^\dagger \hat{a}_\gamma
\]

\[
= \sum_{\alpha, \beta, \gamma, \delta} \langle \alpha | \otimes | \hat{C}_2 | \gamma \otimes \delta \rangle \times \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \equiv \hat{C}.
\]

Q. E. D.