Problem 6.1:
We are interested in a tree-level (in QED) elastic scattering of an electron off a proton via a virtual photon exchange. The amplitude for this process has form

\[ M = \frac{j_\mu(e^-) j^{\mu}(p^+)}{q^2} \]  
(S.1)

where \( q = p' - p = k - k' \) is the momentum of the virtual photon,

\[ j_\mu(e^-) = -e \, \bar{u}(k') \gamma^\mu u(k) \]  
(S.2)

is the electron’s tree-level electromagnetic current while the proton’s current (and hence the vertex) has non-trivial form factors due to strong interactions:

\[ j^{\mu}(p^+) = +e \, \bar{u}(p') \left[ \gamma^\mu F_1(q^2) + \frac{i \sigma^{\mu\nu} q_\nu}{2M} F_2(q^2) \right] u(p) \]
\[ = +e \, \bar{u}(p') \left[ \gamma^\mu (F_1 + F_2)(q^2) - \frac{(p + p')^\mu}{2M} F_2(q^2) \right] u(p). \]  
(S.3)

We need to sum / average the mod-squared amplitude (S.1) over the electron’s and proton’s spin states, thus

\[ \frac{1}{4} \sum_{\text{all spins}} |M|^2 = \frac{1}{4(q^2)^2} \times \sum_{\text{electron's spins}} j^{\mu}(e^-) j^{\nu}(e^-) \times \sum_{\text{proton's spins}} j^{\mu}(p^+) j^{\nu}(p^+) \]
\[ = \frac{e^2}{(q^2)^2} \left[ k'_{\mu} k'_{\nu} + k_{\mu} k'_{\nu} - g_{\mu\nu}(kk') \right] \times \sum_{\text{proton's spins}} j^{\mu}(p^+) j^{\nu}(p^+) \]  
(S.4)

where the second equality — the sum over the electron’s spin states — should be familiar by now; for simplicity, I neglected the electron’s mass in thus sum.
Summing over the proton's spin states is more complicated:

\[
\sum_{\text{proton's spins}} j^\mu(p^+) j^\nu(p^+) = e^2 \text{ tr} \left\{ \gamma^\mu (F_1 + F_2) - \frac{(p + p')^\mu}{2M} F_2 (\not{p} + M) \right. \\
\left. \gamma^\nu (F_1 + F_2) - \frac{(p + p')^\nu}{2M} F_2 (\not{p'} + M) \right\} \\
= e^2 (F_1 + F_2)^2 \text{ tr} \left[ \gamma^\mu (\not{p} + M) \gamma^\nu (\not{p'} + M) \right] + e^2 \frac{F_2^2 (p + p')^\mu(p + p')^\nu}{4M^2} \text{ tr} \left[ (\not{p} + M)(\not{p'} + M) \right] \\
- e^2 (F_1 + F_2)^2 F_2 \left\{ \frac{(p + p')^\mu}{2M} \text{ tr} \left[ (\not{p} + M) \gamma^\nu (\not{p'} + M) \right] + (\mu \leftrightarrow \nu) \right\} \\
= e^2 (F_1 + F_2)^2 \times 4 \left[ (p^\mu p'^\nu + p'^\mu p^\nu + g^{\mu\nu}(M^2 - pp')) \right] + e^2 \frac{F_2^2}{4M^2} \times 4[pp' + M^2] \\
- 2e^2 (F_1 + F_2)^2 \times 2[(p + p')^\mu(p + p')^\nu] \\
\langle \text{using } p^\mu p'^\nu + p'^\mu p^\nu = \frac{1}{2} (p + p')^\mu(p + p')^\nu - \frac{1}{2} q^\mu q^\nu \rangle \\
= e^2 (F_1 + F_2)^2 \times \left[ -2q^\mu q^\nu + 4g^{\mu\nu}(M^2 - pp') \right] + e^2 (p + p')^\mu(p + p')^\nu \times \left[ 2(F_1 + F_2)^2 + F_2^2 \left( 1 + \frac{pp'}{M^2} \right) - 4(F_1 + F_2)^2 \right] \\
\langle \text{using } q^2 = 2M^2 - 2pp' \rangle \\
= 2e^2 (F_1 + F_2)^2 \times \left( q^2 g^{\mu\nu} - q^\mu q^\nu \right) + 2e^2 (p + p')^\mu(p + p')^\nu \times \left( F_1^2 - \frac{q^2}{4M^2} F_2^2 \right). \\
\text{(S.5)}
\]

Substituting eq. (S.5) back into eq. (S.4) now gives us

\[
\frac{1}{2} \sum_{\text{all spins}} |\mathcal{M}|^2 = \frac{2e^4}{(q^2)^2} \left[ \mathcal{A} \left( F_1^2 - \frac{q^2}{4M^2} F_2^2 \right) + \mathcal{B} (F_1 + F_2)^2 \right] \\
\text{(S.6)}
\]
where
\[ \mathcal{A} = [k'_\mu k_\nu + k_\mu k'_\nu - g_{\mu\nu}(kk')] \times (p + p')^\mu(p + p')^\nu \]
\[ = 2(pk + p'k)(pk' + p'k') - (kk')(p + p')^2 \]
or in terms of Mandelstam’s kinematic variables \( s, t \) and \( u \),
\[ = \frac{1}{2}(s - u)^2 + \frac{1}{2}t(4M^2 - t) \]
\[ = 2(s - 2M^2)^2 + 2st \]
and
\[ \mathcal{B} = [k'_\mu k_\nu + k_\mu k'_\nu - g_{\mu\nu}(kk')] \times [q^2 g^{\mu\nu} - q^\mu q^\nu] \]
\[ = (2 - 4 + 1) \times (kk')q^2 - 2(kq)(k'q) \]
\[ = -(\frac{1}{2}t)(t) - 2(\frac{1}{2}t)(-\frac{1}{2}t) = -t^2. \]

In the lab frame,
\[ s = M^2 + 2ME, \quad t = q^2 = -2EE'(1 - \cos \theta) \]
and (cf. discussion of the Compton effect in the textbook)
\[ E' = \frac{ME}{M + E(1 - \cos \theta)}. \]

Also,
\[ \left( \frac{d\sigma}{d\Omega} \right)_{\text{lab}} = \left( \frac{E'}{8\pi ME} \right)^2 \times \frac{1}{4} \sum_{\text{all}} |\mathcal{M}|^2. \]

Combining this formulæ with eq. (S.6) gives us
\[ \left( \frac{d\sigma}{d\Omega} \right)_{\text{lab}} = \frac{\alpha^2}{8M^2E^4(1 - \cos \theta)^2} \times \left[ \mathcal{A} \left( F_1^2 - \frac{q^2}{4M\tau}F_2^2 \right) + \mathcal{B} (F_1 + F_2)^2 \right] \]
where we now evaluate eqs. (S.7) and (S.8) as
\[ \mathcal{A} = \frac{4M^3E^2(1 + \cos \theta)}{M + E(1 - \cos \theta)}, \]
\[ \mathcal{B} = -(q^2)^2 = q^2 \times \frac{2ME^2(1 - \cos \theta)}{M + E(1 - \cos \theta)}. \]
Substituting eqs. (S.13) into eq. (S.12) finally gives us the Rosenbluth formula

\[
\left( \frac{d\sigma}{d\Omega} \right)_{\text{lab}} = \frac{\alpha^2}{2E^2(1 - \cos \theta)^2} \times \frac{(1 + \cos \theta) \left( F_1^2 - \frac{q_e^2}{4\pi M_f} F_2^2 \right) + (1 - \cos \theta) \frac{q_e^2}{2M_f} (F_1 + F_2)^2}{1 + \frac{E}{M_f}(1 - \cos \theta)}.
\]

Q.E.D.

Problem 6.3(a):
The leading-order effect of the Higgs boson on the electron's magnetic moment comes from the following Feynman diagram:

\[
\begin{align*}
\text{In terms of the } e\gamma \text{ vertex correction, we have}
\end{align*}
\]

\[
i e \bar{u}(\delta \Gamma^\mu_{\text{amp}}) u = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_h^2 + i\epsilon} \times \bar{u}(p') \frac{-i\lambda}{\sqrt{2}} \frac{i}{\not{p} + \not{k} - m_e + i\epsilon} \frac{1}{\not{p} + \not{k} - m_e + i\epsilon} \frac{i}{\sqrt{2}} u(p),
\]

which can be evaluated in exactly the same way as the similar QED correction with a virtual photon.

Using Feynman parameters \( x + y + z = 1 \) for the three denominators, we have

\[
\delta \Gamma^\mu_{\text{amp}} = \int dx dy dz \delta(x + y + z - 1) \int \frac{d^4\ell}{(2\pi)^4} \frac{i\lambda^2 N^\mu}{(\ell^2 - \Delta + i\epsilon)^2}
\]

where

\[
\ell = k + xp + yp', \quad \Delta = zm_h^2 + (1 - z)^2 m_e^2 - xyq^2
\]

and

\[
N^\mu = (\not{p}' + \not{k} + m_e) \gamma^\mu (\not{p} + \not{k} + m_e)
\]

\[
\approx \left[ (1 + z)^2 m_e^2 + xyq^2 - \frac{1}{2} \ell^2 \right] \times \gamma^\mu + 2m_e^2(1 - z^2) \times \frac{i\sigma^{\mu\nu}q_\nu}{2m_e^2}.
\]

The second expression here follows from the first after a few tricks you should have learned from
the QED calculation. As in QED, the first term on the second line of eq. (S.18) contributes to the 
\( \delta F_1(q^2) \) while the second term contributes to the \( \delta F_2(q^2) \), and hence to the anomalous magnetic moment. Thus,

\[
\delta_{\text{Higgs}} F_2(q^2) = \int \int \int dxdydz \delta(x + y + z - 1) \left\{ 2m_e^2 \lambda^2 (1 - z^2) \right\} \frac{d^4 \ell}{(2\pi)^4} \frac{i}{(\ell^2 - \Delta + i\epsilon)^3}
\]

\[
\delta_{\text{Higgs}} F_2(q^2) = \frac{\lambda^2}{16\pi^2} \int \int \int dxdydz \delta(x + y + z - 1) \frac{m_e^2(1 - z^2)}{\Delta = zm_h^2 + (1 - z)^2m_e^2 - xyq^2}
\]

and therefore

\[
\delta_{\text{Higgs}} \left( \frac{g - 2}{2} \right) = \delta_{\text{Higgs}} F_2(q^2 = 0)
\]

\[
= \frac{\lambda^2}{16\pi^2} \int \int \int dxdydz \delta(x + y + z - 1) \frac{m_e^2(1 - z^2)}{zm_h^2 + (1 - z)^2m_e^2}
\]

\[
= \frac{\lambda^2}{16\pi^2} \int_0^1 dz \frac{(1 - z)(1 - z^2)m_e^2}{zm_h^2 + (1 - z)^2m_e^2}
\]

\[
= \frac{\lambda^2m_e^2}{16\pi^2m_h^2} \left[ \log \frac{m_h^2}{m_e^2} - \frac{7}{6} + O \left( \frac{m_e^2}{m_h^2} \right) \right].
\]

### Problem 6.3(b):

According to eq. (S.20), the effect of the virtual Higgs boson on the electron’s magnetic moment is suppressed by two very small factors, namely \((m_e/m_h)^2\) and \(\lambda^2 = (m_e/\langle H \rangle)^2\) where \(\langle H \rangle \approx 245\) GeV is the Higgs field’s vacuum expectation value that gives rise to the electron’s mass in the first place. Presently, the Higgs mass \(m_h\) is not known, but there is experimental lower limit \(m_h \gtrsim 100\) GeV. Hence,

\[
\delta_{\text{Higgs}} \left( \frac{g - 2}{2} \right)_e \approx 1.7 \cdot 10^{-23},
\]

\[
\delta_{\text{Higgs}} \left( \frac{g - 2}{2} \right)_\mu \approx 1.7 \cdot 10^{-14},
\]

both correction being much smaller than the present-day experimental uncertainties of the two leptons’ anomalous magnetic moments.
Problem 6.3(c):
The axion's contribution to the anomalous magnetic moment can be calculated similarly to that of the Higgs boson. Again, the $e\gamma$ vertex correction is given by eq. (S.16), only now

$$
\mathcal{N}^\mu = -\gamma^5 (\not{p} + \not{k} + m_e) \gamma^\mu (\not{p} + \not{k} + m_e) \gamma^5 \\
\simeq \left[(1 - z)^2 m_e^2 + x y q^2 - \frac{1}{2} \xi^2\right] \times \gamma^\mu - 2 m_e^2 (1 - z)^2 \times \frac{i \sigma^{\mu \nu} q_\nu}{2 m_e}.
$$

(S.22)

Consequently,

$$
\delta_{\text{axion}} \left( \frac{g - 2}{2} \right) = -\frac{\lambda^2}{16 \pi^2} \int_0^1 dz \frac{(1 - z)^3 m_e^2}{z m_a^2 + (1 - z)^2 m_e^2} = -\frac{\lambda^2}{16 \pi^2} I(m_a^2 / m_e^2),
$$

where the analytic formula for the integral $I(m_a^2 / m_e^2)$ is too complicated to display here. Asymptotically, $I \approx \frac{1}{2}$ for a light axion ($m_a \ll m_e$) and $I \approx \frac{m_a^2}{m_e^2} \left[\log \left(\frac{m_a^2}{m_e^2}\right) - \frac{11}{6}\right] \ll 1$ for a heavy axion ($m_a \gg m_e$).

Given the current experimental limit $|\delta \left( \frac{g - 2}{2} \right)_e| < 10^{-11}$, there is a rather low experimental limit on the electron-axion coupling: A light axion must have $|\lambda| < 4 \cdot 10^{-5}$; for a heavier axion, $|\lambda| < 4 \cdot 10^{-5} / \sqrt{2 I}$.