Problem 1(a):

Let us evaluate the trace of the Casimir operator \( \hat{C}_2 \) over the irreducible representation \( (r) \).

On one hand,

\[
\text{tr}_{(r)} \left( \hat{C}_2 \right) \overset{\text{def}}{=} \sum_a T^a T^a = \sum_a \text{tr}_{(r)} (T^a T^a) = \sum_a (R(r) \times 1) = R(r) \times \text{dim}(G)
\] (S.1)

where \( \text{dim}(G) \overset{\text{def}}{=} \text{dim}(\text{Adj}(G)) \) is the number of the generators of the symmetry group \( G \) — which is also the dimension of the adjoint representation of \( G \), hence the notation. On the other hand,

\[
\text{tr}_{(r)} \left( \hat{C}_2 \right) = \text{tr}_{(r)} \left( \hat{C}_2 \right)|_{(r)} = \text{tr}_{(r)} (C(r) \times 1) = C(r) \times \text{dim}(r).
\] (S.2)

Together, eqs. (S.1) and (S.2) immediately imply eq. (2), Q.E.D.

For the special case of \( G = SU(2) \), we have distinct irreducible representations uniquely specified by the value ‘isospin’ \( I \) and has \( C(I) = I(I + 1) \). Also, \( \text{dim}(I) = (2I + 1) \) and \( \text{dim}(G) = 3 \), hence,

\[
R(I) = C(I) \times \frac{\text{dim}(I)}{\text{dim}(G)} = \frac{1}{3} I(I + 1)(2I + 1).
\] (S.3)

Problem 1(b):

Unlike the Casimir value \( C(r) \), the index \( R(r) \) is well defined for any complete representation \( (r) \), irreducible or otherwise. For a reducible representation

\[
(r) = \bigoplus_{i=1}^{n} (r_i) \equiv (r_1) \oplus (r_2) \oplus \cdots \oplus (r_n)
\]
one clearly has

\[
\text{tr}_{(r)} \left( T^a T^b \right) = \text{tr} \left( T^a T^b \big|_{\oplus_{i=1}^n (r_i)} \right) = \sum_{i=1}^n \text{tr} \left( T^a T^b \big|_{(r_i)} \right)
\]

\[
= \sum_{i=1}^n \left( R(r_i) \times \delta^{ab} \right) = \delta^{ab} \times \sum_{i=1}^n R(r_i)
\]

and thus

\[
R(r) = \sum_{i=1}^n R(r_i).
\]

In particular, for a reducible representation

\[
(r) = \bigoplus_{i=1}^n (I_i)
\]

of the isospin group \( SU(2) \), one has

\[
R(r) = \sum_{i=1}^n \frac{1}{3} I(I + 1)(2I + 1).
\]

Now consider a bigger symmetry group \( G \) which contains the ‘isospin’ \( SU(2) \) as a subgroup. Then any complete representation \( (r) \) of \( G \) is automatically a complete representation of the \( SU(2) \subset G \). Generally, such \( (r) \) would be a reducible representation of the \( SU(2) \) even if it were irreducible from the bigger group \( G \) point of view, thus we expect \( (r) \) to decompose into \( (I_1) \oplus (I_2) \oplus \cdots \oplus (I_n) \) from the \( SU(2) \) point of view. (The isospins \( I_1, I_2, \ldots I_n \) may be all distinct or all equal or whatever.) Consequently, for \( a, b = 1, 2, 3 \ i.e., T^a \text{ and } T^b \) being generators of the \( SU(2) \subset G \), we have

\[
\text{tr}_{(r)} \left( T^a T^b \right) = \delta^{ab} \times \sum_{i=1}^n \frac{1}{3} I(I + 1)(2I + 1),
\]

cf. eq. (S.6).
Now, let us suppose that the Lie group $G$ is simple, that is, all its generators are related to each other by the symmetry $G$ itself. In this case, for any complete representation $(r)$,

$$\text{tr}_{(r)}(T^aT^b) = R(r) \times \delta^{ab} \quad \text{(S.8)}$$

with the same index $R(r)$ for any two generators $T^a$ and $T^b$ of $G$, hence eq. (3), $\Box$. Q.E.D.

**Problem 1(c):**

From the $SU(2) \subset SU(N)$ point of view, the fundamental representation $\mathbf{N}$ of the $SU(N)$ decomposes into a doublet plus $(N - 2)$ singlets,

$$\mathbf{N} = 2 + (N - 2) \times 1 \equiv (I = \frac{1}{2}) + (N - 2) \times (I = 0) \quad \text{(S.9)}$$

hence according to eq. (3),

$$R(\mathbf{N}) = R(I = \frac{1}{2}) + (N - 2) \times R(I = 0) = \frac{1}{2} + (N - 2) \times 0 = \frac{1}{2}$$

and consequently

$$C(\mathbf{N}) = R(\mathbf{N}) \times \frac{\text{dim}(G)}{\text{dim}(\mathbf{N})} = \frac{1}{2} \times \frac{N^2 - 1}{N} = \frac{N^2 - 1}{2N} \quad \text{(4)}$$

Now consider the adjoint representation of the $SU(N)$. Let us form a tensor product of the fundamental representation $\mathbf{N}$ and the conjugate (anti-fundamental) representation $\overline{\mathbf{N}}$.

Given the transformation laws

$$\Psi \rightarrow U\Psi, \quad \text{i.e.} \quad \Psi'_j = U^k_j \Psi_k,$$

$$\overline{\Psi} \rightarrow \overline{\Psi} U^\dagger, \quad \text{i.e.} \quad \overline{\Psi'}^\ell = \overline{\Psi}^m U^*_{m\ell},$$

it follows that the tensor product is an $N \times N$ matrix $\Phi^k_j$ which transforms according to

$$\Phi' = U\Phi U^\dagger \quad \text{i.e.} \quad \Phi'^\ell_j = U^k_j \Phi^m_k U^*_{m\ell}. \quad \text{(5)}$$

Hence, forms a reducible representation of the $SU(N)$, namely the tensor sum of the adjoint representation (the traceless part of $\Phi$) plus the trivial singlet representation (the trace...
tr(Φ)). In other words,

\[ N \otimes \overline{N} = \text{Adj} \oplus 1 \quad \text{(S.10)} \]

From the SU(2) ⊂ SU(N) point of view, both the fundamental and the anti-fundamental representations of the SU(N) decompose according to eq. (S.9). (In SU(2), \( \overline{2} = 2 \).) Therefore,

\[
[\text{Adj} + 1]_{SU(N)} = [N \otimes \overline{N}]_{SU(N)} = [2 + (N-2) \times 1]_{SU(2)} \otimes [2 + (N-2) \times 1]_{SU(2)} = [3 + 1 + 2(N-2) \times 2 + (N-2)^2 \times 1]_{SU(2)},
\]

i.e. \([\text{Adj}]_{SU(N)} = [3 + 2(N-2) \times 2 + (N-2)^2 \times 1]_{SU(2)} \),

(S.11)

and consequently

\[
R(\text{Adj}) = I(3) + 2(N-2)I(2) + (N-2)^2I(1) = 2 + 2(N-2) \times \frac{1}{2} + (N-2)^2 \times 0 = N.
\]

(S.12)

Finally,

\[
C(G) \overset{\text{def}}{=} C(\text{Adj}(G)) = R(\text{Adj}) \times \frac{\text{dim}(G)}{\text{dim}(G)} = R(\text{Adj}) = N.
\]

(S.13)

Problem 1(d):
Consider the two-index symmetric tensor \( S_{(ij)} \) representation of the SU(N) symmetry group. Denote the index \( i = \alpha \) if \( i = 1,2 \) or \( i = \mu \) if \( i = 3,4,\ldots,N \) and likewise \( j = \beta \) if \( j = 1,2 \) and \( j = \nu \) if \( j = 3,4,\ldots,N \). Thus, the complete set of independent \( S_{(ij)} \) decomposes into \( S_{(\alpha\beta)}, S_{\alpha\mu}, S_{(\mu\nu)} \). The SU(2) ⊂ SU(N) acts on indices \( \alpha, \beta = 1,2 \) and ignores indices \( \mu, \nu = 3,4,\ldots,N \). Hence, from the SU(2) point of view, the symmetric tensor of the SU(N) decomposes into one 2-index symmetric tensor \( S_{(\alpha\beta)} \), plus \( (N-2) \) doublets \( S_{\alpha\mu} \),
plus \((N-1)(N-2)/2\) singlets \(S_{(\mu\nu)}\). Consequently,

\[
R(S) = R(3) + (N-2)R(2) + \frac{1}{2}(N-1)(N-2)R(1) = 2 + (N-2) \times \frac{1}{2} + 0 = \frac{1}{2}(N+2),
\]

and hence

\[
C(S) = R(S) \times \frac{\dim(G)}{\dim(S)} = \frac{N+2}{2} \times \frac{N^2 - 1}{2N(N+1)} = \frac{N^2 + N - 2}{N}.
\]  

(S.14)

Similarly, the two-index anti-symmetric tensor \(A_{[ij]}\) decomposes into one 2-index anti-symmetric \(SU(2)\) tensor \(A_{[\alpha\beta]}\), plus \((N-2)\) \(SU(2)\) doublets \(A_{\alpha,\mu}\), plus \((N-2)(N-3)/2\) \(SU(2)\) singlets \(A_{[\mu\nu]}\). Furthermore, the 2-index anti-symmetric tensor \(A_{[\alpha\beta]}\) of the \(SU(2)\) is equivalent to the trivial singlet \(A \times \epsilon_{[\alpha\beta]}\), therefore

\[
(A) = (N-2) \times 2 + \text{singlets}
\]

and hence

\[
R(A) = (N-2) \times \frac{1}{2} + 0 = \frac{1}{2}(N-2)
\]  

(S.16)

and

\[
C(A) = R(A) \times \frac{\dim(G)}{\dim(A)} = \frac{N-2}{2} \times \frac{N^2 - 1}{2N(N-1)} = \frac{N^2 - N - 2}{N}.
\]  

(S.17)

Problem 2(a):
For an arbitrary gauge transform \(U(x)\) we have

\[
\Phi'(x) = U(x)\Phi(x)U^\dagger(x), \quad A'_\mu(x) = U(x)A_\mu(x)U^\dagger(x) - i(\partial_\mu U(x))U^\dagger(x),
\]

and therefore — according to eq. (8) —

\[
D_\mu'\Phi'(x) = \partial_\mu\Phi'(x) - i[A'_\mu(x), \Phi'(x)] = \partial_\mu(U\Phi U^\dagger) - i[U(A_\mu - i(U^\dagger\partial_\mu U))U^\dagger, U\Phi U^\dagger]
\]

\[
= U(\partial_\mu \Phi + [(U^\dagger\partial_\mu U), \Phi])U^\dagger - iU[A_\mu - i(U^\dagger\partial_\mu U), \Phi]U^\dagger
\]

\[
= U(\partial_\mu \Phi - i[A_\mu, \Phi])U^\dagger
\]

\[
\equiv U(x)\left(D_\mu \Phi(x)\right)U^\dagger(x).
\]
In other words, the covariant derivative defined by eq. (8) is indeed covariant.

Problem 2(b):

\[ D_\mu D_\nu \Phi = D_\mu (\partial_\nu \Phi - i[A_\nu, \Phi]) = \partial_\mu (\partial_\nu \Phi - i[A_\nu, \Phi]) - i[A_\mu, (\partial_\nu \Phi - i[A_\nu, \Phi])] \\
= \partial_\mu \partial_\nu \Phi - i[(\partial_\mu A_\nu), \Phi] - i[A_\nu, \partial_\mu \Phi] - i[A_\mu, \partial_\nu \Phi] - [A_\mu, [A_\nu, \Phi]]. \]  
(S.19)

Similarly,

\[ D_\nu D_\mu \Phi = \partial_\nu \partial_\mu \Phi - i[(\partial_\nu A_\mu), \Phi] - i[A_\mu, \partial_\nu \Phi] - i[A_\nu, \partial_\mu \Phi] - [A_\nu, [A_\mu, \Phi]]. \]  
(S.20)

The difference between these two formulas cancels several terms living behind

\[ [D_\mu, D_\nu] \Phi = -i[(\partial_\mu A_\nu), \Phi] + i[(\partial_\nu A_\mu), \Phi] - [A_\mu, [A_\nu, \Phi]] + [A_\nu, [A_\mu, \Phi]] \\
= -i[(\partial_\mu A_\nu - \partial_\nu A_\mu), \Phi] - [[A_\mu, A_\nu], \Phi] \]  
(S.21)

\[ \equiv -i[F_{\mu\nu}, \Phi]. \]

Problem 2(c):

\[ D_\lambda F_{\mu\nu} = \partial_\lambda (\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]) - i[A_\lambda, (\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu])] \\
= (\partial_\lambda \partial_\mu A_\nu - \partial_\lambda \partial_\nu A_\mu) - i[(A_\mu, \partial_\lambda A_\nu) - [A_\nu, \partial_\lambda A_\mu]) - i[(A_\mu, A_\nu) - [A_\lambda, A_\nu] - [A_\lambda, A_\mu]]. \]  
(S.22)

For each group of terms here, summing over cyclic permutation of the Lorentz indices \( \lambda \rightarrow \mu \rightarrow \nu \rightarrow \lambda \) produces a zero:

\( (\partial_\lambda \partial_\mu A_\nu - \partial_\lambda \partial_\nu A_\mu) + (\partial_\mu \partial_\nu A_\lambda - \partial_\mu \partial_\lambda A_\nu) + (\partial_\nu \partial_\lambda A_\mu - \partial_\nu \partial_\mu A_\lambda) = 0, \)

\( ([A_\mu, \partial_\lambda A_\nu] - [A_\nu, \partial_\lambda A_\mu]) + ([A_\nu, \partial_\mu A_\lambda] - [A_\lambda, \partial_\mu A_\nu]) + ([A_\lambda, \partial_\nu A_\mu] - [A_\mu, \partial_\nu A_\lambda]) = 0, \)

\( ([A_\lambda, \partial_\mu A_\nu] - [A_\nu, \partial_\mu A_\lambda]) + ([A_\mu, \partial_\nu A_\lambda] - [A_\lambda, \partial_\nu A_\mu]) + ([A_\lambda, \partial_\nu A_\mu] - [A_\mu, \partial_\nu A_\lambda]) = 0, \)

\( [A_\lambda, [A_\mu, A_\nu]] + [A_\mu, [A_\nu, A_\lambda]] + [A_\nu, [A_\lambda, A_\mu]] = 0. \)  
(S.23)

and consequently

\[ D_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} = 0. \]  
(9)

Problem 2(d):
Let \( A^\mu(x) \rightarrow A^\mu(x) + \delta A^\mu(x) \); then the first variation of \( F^{\mu\nu}(x) \) is
\[
\delta_a (F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - i[A^\mu, A^\nu]) = \partial^\mu \delta A^\nu - \partial^\nu \delta A^\mu - i[A^\mu, \delta A^\nu] - i[\delta A^\mu, A^\nu] = D^{\mu} \delta A^\nu - D^{\nu} \delta A^\mu
\]
and consequently,
\[
\delta_1 \mathcal{L}_{YM} = \frac{2}{g^2} \text{tr} (F_{\mu\nu} D^{\mu} \delta A^\nu).
\] (S.24)

Next, consider a total derivative of general form
\[
\text{tr}((D^{\mu}B)C) + \text{tr}(B(D^{\mu}C)) = \text{tr}(D^{\mu}(BC)) = \text{tr}(\partial^{\mu}(BC) - i[A^\mu, BC]) = \partial^{\mu}(\text{tr}(BC)) + 0,
\]
which allows us to integrate by parts traces involving gauge-covariant derivatives. Thus, eq. (S.25) can be written as
\[
\delta_1 \mathcal{L}_{YM} = \frac{2}{g^2} \text{tr} ((D^{\mu}F_{\mu\nu})\delta A^\nu) + \partial^{\mu} \{ \cdots \} = \frac{1}{g^2} (D^{\mu}F_{\mu\nu})^a \delta A^{a\nu} + \partial^{\mu} \{ \cdots \}
\]
and consequently
\[
\frac{\delta}{\delta A^{a\nu}(x)} \left[ S_{YM} = \int \mathcal{L}_{YM} d^4x \right] = \frac{1}{g^2} (D^{\mu}F_{\mu\nu})^a,
\]
which immediately implies the classic field equation of motion
\[
D^{\mu}F_{\mu\nu} = 0.
\] (S.29)

**Problem 2(e):**
Eq. (12) follows immediately from the previous result and writing the new Lagrangian (11) as
\[
\mathcal{L} = \mathcal{L}_{YM} + \overline{\Psi}(i \partial - m)\Psi + A^{a\nu}J^a_\nu.
\] (S.30)

**Problem 2(f):**
\[
D^\nu J^\nu \propto D^\nu (D_\mu F^{\mu\nu}) = -\frac{1}{2}[D_\mu, D_\nu]F^{\mu\nu} = \frac{i}{2} [F_{\mu\nu}, F^{\mu\nu}] = 0.
\] (S.31)
where the first equality follows from \( F^{\mu\nu} = -F^{\nu\mu} \), the second from previous result (b), and the third from the fact that \( F^{\mu\nu} \) commutes with itself.
Problem 2(g):

The Dirac equations following from the Lagrangian (11) are

\[(i\bar{D} - m)\Psi = 0, \quad \bar{\Psi}(-i\bar{D} - m) = 0,\]

or

\[i\gamma^\mu \partial_\mu \Psi + A^a_{\mu}\gamma^\mu T^a \Psi - m\Psi = 0, \quad -i\partial_\mu \bar{\Psi}\gamma^\mu + A^a_{\mu}\bar{\Psi}\gamma^\mu T^a - m\bar{\Psi} = 0.\]  \hspace{1cm} (S.32)

Consequently, the fermionic current

\[J^{a\nu} = \bar{\Psi}\gamma^\nu T^a \Psi \hspace{1cm} (13)\]

(for fermions in the fundamental representation of the gauge group \(T^a = \frac{1}{2} \lambda^a\)) satisfies

\[\partial_\nu J^{a\nu} = (\partial_\nu \bar{\Psi}\gamma^\nu) T^a \Psi + \bar{\Psi} T^a (\gamma^\nu \partial_\nu \Psi)
\[= \bar{\Psi} (im - i\gamma^\nu T^b A^b_{\nu}) T^a \Psi + \bar{\Psi} T^a (-im + i\gamma^\nu T^b A^b_{\nu}) \Psi
\[= -iA^b_{\nu} \times \bar{\Psi} \gamma^\nu [T^b, T^a] \Psi = -f^{abc} A^b_{\nu} \bar{\Psi} \gamma^\nu T^c \Psi
\[\equiv -f^{abc} A^b_{\nu} J^{c\nu},\]  \hspace{1cm} (S.33)

and therefore

\[D_\nu J^{a\nu} = \partial_\nu J^{a\nu} + f^{abc} A^b_{\nu} J^{c\nu} = 0.\]  \hspace{1cm} (S.34)

Problem 2(h):

Finally, consider the second variation of the YM action. The first variation of the YM tension \(F^{\mu\nu}\) is given by eq. (S.24) while the second variation is simply

\[\delta_2 F^{\mu\nu} = -i[\delta A^\mu, \delta A^\nu].\]  \hspace{1cm} (S.35)

Therefore,

\[\delta_2 \left[ \frac{1}{2} \text{tr} (F_{\mu\nu} F^{\mu\nu}) \right] = \frac{1}{2} \text{tr} (\delta F_{\mu\nu} \delta_1 F^{\mu\nu}) + \text{tr} (F_{\mu\nu} \delta_2 F^{\mu\nu})
\[= \frac{1}{2} \text{tr} \left( (D^\mu \delta A^\nu - D^\nu \delta A^\mu)^2 \right) - i \text{tr} (F_{\mu\nu} [\delta A^\mu, \delta A^\nu])
\[= \text{tr} ((D^\mu \delta A^\nu)(D^\mu \delta A^\nu)) - \text{tr} ((D_\mu \delta A^\nu)(D^\nu \delta A^\mu)) - 2i \text{tr} (F_{\mu\nu} \delta A^\mu \delta A^\nu),\]  \hspace{1cm} (S.36)
which may be further simplified by discarding total derivatives à la eq. (S.26) in anticipation of integration \( \int d^4x \). Indeed, the first term on the last line of eq. (S.36) becomes

\[
\text{tr} \left( (D_\mu \delta A_\nu)(D^\mu \delta A^\nu) \right) = \partial_\mu (\cdots) - \text{tr} \left( \delta A_\nu D^2 \delta A^\nu \right), \tag{S.37}
\]

while for the second term we have

\[
- \text{tr} \left( (D_\mu \delta A_\nu)(D^\nu \delta A^\mu) \right) = \partial_\mu (\cdots) + \text{tr} \left( \delta A_\nu D_\mu D^\nu \delta A^\mu \right) = \partial_\mu (\cdots) + \text{tr} \left( \delta A_\nu D_\nu D_\mu \delta A^\mu \right) + \text{tr} \left( \delta A_\nu [D^\mu, D^\nu] \delta A_\mu \right) = \partial_\mu (\cdots) - \text{tr} \left( (D^\nu \delta A^\nu)^2 \right) - 2i \text{tr} \left( F^{\mu \nu} \delta A_\mu \delta A_\nu \right). \tag{S.38}
\]

Consequently,

\[
\delta_2 \left[ \frac{1}{2} \text{tr} \left( F_{\mu \nu} F^{\mu \nu} \right) \right] = - \text{tr} \left( \delta A_\nu D^2 \delta A^\nu \right) - \text{tr} \left( (D^\nu \delta A^\nu)^2 \right) - 4i \text{tr} \left( F^{\mu \nu} \delta A_\mu \delta A_\nu \right) + \partial_\mu (\cdots) \tag{S.39}
\]

and hence

\[
\delta_2 \left[ \int d^4x \mathcal{L}_{YM} \right] = \frac{1}{g^2} \int d^4x \left[ \text{tr} \left( \delta A^\mu D^2 \delta A_\mu \right) + \text{tr} \left( (D_\mu \delta A^\mu)^2 \right) + 4i \text{tr} \left( F_{\mu \nu} \delta A^\mu \delta A_\nu \right) \right]. \tag{S.40}
\]

Q.E.D.