1. Consider the matrix $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$.

(a) Show that $\gamma^5$ anticommutes with each of the $\gamma^\mu$ matrices, $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$.

(b) Show that $\gamma^5$ is hermitian and that $(\gamma^5)^2 = 1$.

(c) Show that $\gamma^5 = (-i/24)\epsilon_{\kappa\lambda\mu\nu}\gamma^{\kappa}\gamma^{\lambda}\gamma^{\mu}\gamma^{\nu}$ and $\gamma^{[\kappa\gamma^{\lambda}\gamma^{\mu}\gamma^{\nu}]} = -i\epsilon^{\kappa\lambda\mu\nu}\gamma^5$.

(Sign convention: $\epsilon^{0123} = +1, \epsilon_{0123} = -1$.)

(d) Show that $\gamma^{[\lambda\gamma^{\mu}\gamma^{\nu}]} = i\epsilon^{\kappa\lambda\mu\nu}\gamma_{\kappa}\gamma^5$.

(e) Show that any $4 \times 4$ matrix $\Gamma$ is a unique linear combination of the following 16 matrices: $1, \gamma^\mu, \gamma^{[\mu\gamma^{\nu}]}, \gamma^5\gamma^\mu$ and $\gamma^5$.

Under continuous Lorentz symmetries, a Dirac spinor field $\Psi(x)$ transforms according to

$\Psi'(x') = M(L)\Psi(x = L^{-1}x')$ where $M(L = e^\theta) = \exp(-i\frac{\theta}{2}\gamma_{\alpha\beta}S_{\alpha\beta})$. Consider the transformation rules for the independent bilinear products $\Psi\Gamma\Psi$ of a Dirac field and its conjugate $\overline{\Psi}(x)$, namely (cf. (e))

\begin{equation}
S = \overline{\Psi}\Psi, \quad V^\mu = \overline{\Psi}\gamma^\mu\Psi, \quad T^{\mu\nu} = \overline{\Psi}\gamma^{[\mu\gamma^{\nu}]\Psi}, \quad A^\mu = \overline{\Psi}\gamma^5\gamma^\mu\Psi \quad \text{and} \quad P = \overline{\Psi}\gamma^5\Psi.
\end{equation}

(f) Show that under continuous Lorentz symmetries, the $S$ and the $P$ transform as scalars, the $V^\mu$ and the $A^\mu$ as vectors and the $T^{\mu\nu}$ as an antisymmetric tensor.

2. Under the parity symmetry $\mathcal{P} : (x, t) \mapsto (-x, t)$, Dirac spinor fields transform according to

$\hat{\mathcal{P}} \hat{\Psi}(x, t) \hat{\mathcal{P}} \equiv \hat{\Psi}'(x, t) = \pm\gamma^0 \hat{\Psi}(-x, t)$

where the overall sign depends on the so-called intrinsic parity of a particular Dirac field.

Note: $\hat{\mathcal{P}}$ here is a unitary operator in the fermionic Fock space; by nature of the parity symmetry, $\mathcal{P}^2 = 1$.

(a) Verify the covariance of the Dirac equation under this symmetry.
(b) Find the transformation rules of the bilinears (1) under parity and show that while $S$ is a true scalar and $V$ is a true (polar) vector, $P$ is a pseudoscalar and $A$ is an axial vector.

3. Next, consider the charge-conjugation properties of Dirac bilinears $\bar{\Psi}\Gamma\Psi$. To avoid operator ordering problems, take $\Psi(x)$ and $\Psi^\dagger(x)$ to be “classical” fermionic fields which anticommute with each other, $\Psi_\alpha\Psi^\dagger\beta = -\Psi^\dagger\beta\Psi_\alpha$.

(a) Show that $\hat{C}\bar{\Psi}\Gamma\hat{C} = \bar{\Psi}\Gamma^c\hat{C}$ where $\Gamma^c = \gamma^0\gamma^2\Gamma^\top\gamma^0\gamma^2$.

(b) Calculate $\Gamma^c$ for all 16 independent matrices $\Gamma$ and find out which Dirac bilinears are $\mathcal{C}$–even and which are $\mathcal{C}$–odd.

4. In theories involving both bosons and fermions, one often has to combine commutation and anti-commutation relations of various operators, depending on the overall statistics of the operators involved. For that purpose, it is useful to define a ‘mixed’ commutator bracket

\[
\{\hat{A}, \hat{B}\} \overset{\text{def}}{=} \hat{A}\hat{B} - (-1)^{AB}\hat{B}\hat{A}
\]

where $(-1)^{AB}$ is $-1$ if both $\hat{A}$ and $\hat{B}$ have overall Fermi statistics (i.e., each comprises an odd number of fermionic creation/annihilation operators — the number of bosonic creation/annihilation operators does not matter) and $+1$ in all other cases.

(a) Verify the Leibniz rule for the mixed brackets: $\{\hat{A}, \hat{B}\hat{C}\} = \{\hat{A}, \hat{B}\}\hat{C} + (-1)^{AB}\hat{B}\{\hat{A}, \hat{C}\}$ and write down a similar rule for the $\{\hat{A}\hat{B}, \hat{C}\}$.

(b) Similarly, express $\{\hat{A}\hat{B}, \hat{C}\hat{D}\}$ in terms of appropriate mixed brackets of $\hat{A}$ or $\hat{B}$ with $\hat{C}$ or $\hat{D}$.

(c) Prove the ‘mixed’ Jacobi identity

\[
(-1)^{CA}\{\hat{A}, [\hat{B}, \hat{C}]\} + (-1)^{AB}\{\hat{B}, [\hat{C}, \hat{A}]\} + (-1)^{BC}\{\hat{C}, [\hat{A}, \hat{B}]\} = 0.
\]
In other words (and notations),
\[
[\hat{B}_1, [\hat{B}_2, \hat{B}_3]] + [\hat{B}_2, [\hat{B}_3, \hat{B}_1]] + [\hat{B}_3, [\hat{B}_1, \hat{B}_2]] = 0,
\]
\[
[\hat{B}_1, [\hat{B}_2, \hat{F}]] + [\hat{B}_2, [\hat{F}, \hat{B}_1]] + [\hat{F}, [\hat{B}_1, \hat{B}_2]] = 0,
\]
\[
\{\hat{F}_1, [\hat{F}_2, \hat{B}]\} - [\hat{F}_2, [\hat{B}, \hat{F}_1]] + [\hat{B}, \{\hat{F}_1, \hat{F}_2\}] = 0,
\]
\[
[\hat{F}_1, \{\hat{F}_2, \hat{F}_3\}] + [\hat{F}_2, \{\hat{F}_3, \hat{F}_1\}] + [\hat{F}_3, \{\hat{F}_1, \hat{F}_2\}] = 0,
\]
where 'B' and 'F' indicate the overall statistics of the operator involved.

5. Finally, an exercise in fermionic creation and annihilation operators and their anticommutation relations,
\[
\{a^\dagger_\alpha, a^\dagger_\beta\} = \{a^\dagger_\alpha, a^\dagger_\beta\} = 0, \quad \{a^\dagger_\alpha, a^\dagger_\beta\} = \delta_{\alpha\beta}.
\]

(a) Calculate the commutators \([\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta]\), \([\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta]\) and \([\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta]\).

(b) Consider two one-body operators \(\hat{A}_1\) and \(\hat{B}_1\) and let \(\hat{C}_1\) be their commutator, \(\hat{C}_1 = [\hat{A}_1, \hat{B}_1]\). Let \(\hat{A}\) be the second-quantized forms of \(\hat{A}_\text{tot}\),
\[
\hat{A} = \sum_{\alpha,\beta} \langle \alpha | \hat{A}_1 | \beta \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta,
\]
and ditto for the second-quantized \(\hat{B}\) and \(\hat{C}\).

Verify that \([\hat{A}, \hat{B}] = \hat{C}\).

(c) Calculate the commutator \([\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\alpha \hat{a}_\beta \hat{a}_\gamma \hat{a}_\delta]\).

(d) The second quantized form of a two-body additive operator
\[
\hat{B}_\text{tot} = \frac{1}{2} \sum_{i \neq j} \hat{B}_2(i^\text{th} \text{ and } j^\text{th} \text{ particles})
\]
acting on identical fermions is
\[
\hat{B} = \frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} \langle \alpha \otimes \beta | \hat{B}_2 | \gamma \otimes \delta \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta.
\]

This expression is similar to its bosonic counterpart, but note the reversed order of the annihilation operators \(\hat{a}_\delta\) and \(\hat{a}_\gamma\).
Consider a one-body operator $\hat{A}_1$ and two two-body operators $\hat{B}_2$ and $\hat{C}_2$. Show that if $\hat{C}_2 = \left[ \hat{A}_1(1^{st}) + \hat{A}_1(2^{nd}) \right], \hat{B}_2$, then the respective second-quantized operators in the fermionic Fock space satisfy $\hat{C} = [\hat{A}, \hat{B}]$. 