Problem 1(a):
As explained in class,

\[
\begin{pmatrix}
\sqrt{E - p\sigma} \xi_s \\
\sqrt{E + p\sigma} \xi_s
\end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix}
\xi^\dagger \sqrt{E + p\sigma} \\
\xi^\dagger \sqrt{E - p\sigma}
\end{pmatrix}
\]

(S.1)

where \( \xi_s \) is the ordinary 3D 2–component spinor normalized to \( \xi^\dagger \xi = 1 \) and therefore

\[
\sum_s (\xi_s \xi^\dagger_s) = 1
\]

(S.2)
as a 2 \( \times \) 2 matrix. Consequently, in 4 \( \times \) 4 matrix notations, we have

\[
\sum_s u(p, s) \bar{u}(p, s) = \sum_s \begin{pmatrix}
\sqrt{E - p\sigma} (\xi_s \xi^\dagger_s) \sqrt{E + p\sigma} & \sqrt{E - p\sigma} (\xi_s \xi^\dagger_s) \sqrt{E - p\sigma} \\
\sqrt{E + p\sigma} (\xi_s \xi^\dagger_s) \sqrt{E + p\sigma} & \sqrt{E + p\sigma} (\xi_s \xi^\dagger_s) \sqrt{E - p\sigma}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\sqrt{E^2 - (p\sigma)^2} & (E - p\sigma) \\
(E + p\sigma) & \sqrt{E^2 - (p\sigma)^2}
\end{pmatrix}
\]

(S.3)

\[
= \begin{pmatrix}
m & (E - p\sigma) \\
(E + p\sigma) & m
\end{pmatrix} = m + \not{p}.
\]

Likewise, for the negative-frequency spinors we have

\[
v(p, s) = \begin{pmatrix}
+\sqrt{E - p\sigma} \eta_s \\
-\sqrt{E + p\sigma} \eta_s
\end{pmatrix} \quad \Rightarrow \quad \bar{v}(p, s) = \begin{pmatrix}
-\eta^\dagger \sqrt{E + p\sigma} \\
+\eta^\dagger \sqrt{E - p\sigma}
\end{pmatrix}
\]

(S.4)

and therefore

\[
\sum_s v(p, s) \bar{v}(p, s) = \sum_s \begin{pmatrix}
-\sqrt{E - p\sigma} (\xi_s \xi^\dagger_s) \sqrt{E + p\sigma} & \sqrt{E - p\sigma} (\xi_s \xi^\dagger_s) \sqrt{E - p\sigma} \\
\sqrt{E + p\sigma} (\xi_s \xi^\dagger_s) \sqrt{E + p\sigma} & -\sqrt{E + p\sigma} (\xi_s \xi^\dagger_s) \sqrt{E - p\sigma}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\sqrt{E^2 - (p\sigma)^2} & (E - p\sigma) \\
(E + p\sigma) & -\sqrt{E^2 - (p\sigma)^2}
\end{pmatrix}
\]

(S.5)

\[
= \begin{pmatrix}
-m & (E - p\sigma) \\
(E + p\sigma) & -m
\end{pmatrix} = -m + \not{p}.
\]
Problem 1(b):

The constant spinors \(u \equiv u(p, s)\) and \(\bar{u}' \equiv \bar{u}(p', s')\) satisfy Dirac equations \(\slashed{p}u = mu\) and \(\bar{u}' \bar{\gamma} = m\bar{u}'\). Applying both equations to the Dirac “sandwich” \(\bar{u}'\gamma^\mu u\), we have

\[
\bar{u}'\gamma^\mu u = \frac{1}{m} \bar{u}' \slashed{p}' \times \gamma^\mu u = \frac{1}{m} \bar{u}'\gamma^\mu \times \slashed{p}u = \frac{1}{2m} \bar{u}'(\bar{\gamma}^\mu + \gamma^\mu \slashed{p})u. \tag{S.6}
\]

Furthermore,

\[
\slashed{p}'\gamma^\mu + \gamma^\mu \slashed{p} \equiv p'_\nu \gamma^\nu \gamma^\mu + p_\nu \gamma^\nu \gamma^\mu = \frac{1}{2}(p' + p)_\nu \{\gamma^\nu, \gamma^\mu\} + \frac{1}{2}(p' - p)_\nu [\gamma^\nu, \gamma^\mu] \tag{S.7}
\]

and therefore

\[
\bar{u}'\gamma^\mu u = \frac{(p' + p)^\mu}{2m} \bar{u}' u + \frac{i(p' - p)_\nu}{m} \bar{u}' S^\mu\nu u. \tag{2}
\]

Q.E.D.

Problem 1(c):

The negative-frequency spinors \(v \equiv v(p, s)\) and \(\bar{v}' \equiv \bar{v}(p', s')\) satisfy Dirac equations \(\slashed{p}v = -mv\) and \(\bar{v}' \bar{\gamma} = -mv'\). Consequently, proceeding exactly as above modulo signs, we have

\[
\bar{v}'\gamma^\mu v = \frac{(p' - p)^\mu}{2m} \bar{v}' v + \frac{i(p' + p)_\nu}{m} \bar{v}' S^\mu\nu v,
\]

\[
\bar{v}'\gamma^\mu u = \frac{(-p' + p)^\mu}{2m} \bar{v}' u + \frac{i(-p' + p)_\nu}{m} \bar{v}' S^\mu\nu u, \tag{S.8}
\]

\[
\bar{v}'\gamma^\mu v = \frac{(-p' - p)^\mu}{2m} \bar{v}' v + \frac{i(-p' + p)_\nu}{m} \bar{v}' S^\mu\nu v.
\]

Problem 2(a):

First, let us rewrite the Lorentz algebra in terms of 3–vectors \(\hat{J}\) and \(\hat{K}\):

\[
[J^i, J^j] = i\epsilon^{ij\ell} J^\ell, \quad [J^i, K^j] = i\epsilon^{ij\ell} K^\ell, \quad [K^i, K^j] = -i\epsilon^{ij\ell} J^\ell. \tag{S.9}
\]

Consequently, for the \(\hat{J}_\pm = \frac{1}{2}(\hat{J} \pm i\hat{K})\), we have

\[
[J^i_\pm, J^j_\pm] = i\epsilon^{ij\ell} J^\ell \mp \frac{i}{4}\epsilon^{ij\ell} K^\ell \mp \frac{1}{4}\epsilon^{ij\ell} K^\ell \pm \frac{i}{4}\epsilon^{ij\ell} J^\ell = i\epsilon^{ij\ell} J^\ell_\pm
\]

2
while
\[
\begin{bmatrix}
\hat{j}_1, \hat{j}_2
\end{bmatrix} = \frac{i}{4} \epsilon_{ij\ell} \hat{j}_j \mp \frac{1}{4} \epsilon_{ij\ell} \hat{k}_\ell \pm \frac{1}{4} \epsilon_{ij\ell} \hat{k}_\ell - \frac{i}{4} \epsilon_{ij\ell} \hat{j}_j = 0.
\]

Q.E.D.

Problem 2(b):

First, note the hermiticity of the $\sigma^\mu$ matrices and the fact that any hermitian $2 \times 2$ matrix is a unique linear combination of the four $\sigma_\nu$ with real coefficients. Consequently,

\[
\forall M : M\sigma^\mu M^\dagger = \sigma^\nu L^\nu_\mu(M) \implies X'_\nu = L^\nu_\mu(M)X_\mu
\]

for some real $4 \times 4$ matrix $L^\nu_\mu(M)$. Furthermore, for $M \in SL(2, \mathbb{C})$, i.e. for $\text{det}(M) = 1$, this $L^\nu_\mu(M)$ matrix defines a Lorentz transform for which $X'_\mu X^\mu = X_\mu X^\mu$. To see this, we note that

\[
\det(X_\mu \sigma^\mu) = \det\left(\begin{array}{cccc} X_0 & X_3 & X_1 & -iX_2 \\
X_1 & X_0 & -iX_2 & X_3 
\end{array}\right) = (X_0)^2 - (X_3)^2 - (X_1)^2 - (X_2)^2 \equiv X^2
\]

and then calculate

\[
X'^2 = \det(X'_\mu \sigma^\mu) = \det(M(X_\mu \sigma^\mu)M^\dagger) = |\text{det}(M)|^2 \times \det(X_\mu \sigma^\mu) = 1 \times X^2.
\]

Also, the Lorentz transform $X_\mu \rightarrow X'_\mu = L^\nu_\mu X_\nu$ is orthochronous because

\[
L^0_0 = \frac{1}{2} \text{tr}(\sigma^\nu L^\nu_\mu) = \frac{1}{2} \text{tr}(M\sigma^0 M^\dagger) = \frac{1}{2} \text{tr}(MM^\dagger) > 0.
\]

Problem 2(b*):

The simplest proof the $L^\nu_\mu(M)$ is a proper Lorentz transform involves the group law (problem 2(c) below) and the explicit examples of a pure rotation and a pure boost (problem 2(d) below, eqs. (S.16) and (S.18)), both of which are manifestly proper.

For any $SL(2, \mathbb{C})$ matrix $M$ we may decompose $M = HU$ where $H = \sqrt{MM^\dagger}$ is hermitian and $U = H^{-1}M$ is unitary. (Proof: $UU^\dagger = H^{-1}MM^\dagger H^{-1} = H^{-1}H^2H^{-1} = 1$.) Furthermore, both $H$ and $U$ are unimodular ($\text{det}(H) = \text{det}(U) = 1$), or in other words $H, U \in SL(2, \mathbb{C})$, which
allows us to define two separate Lorentz transforms \( L(H) \) and \( L(U) \). According to the group law, together these two transform accomplish the \( L(M) \) transform,

\[
L(M) = L(H) \times L(U). \tag{S.14}
\]

Now, \( H \) is hermitian, unimodular, and positive definite, hence it has a well-defined logarithm which is hermitian and traceless, \( \text{tr}(\log H) = 0 \). For the \( 2 \times 2 \) matrices, this means \( \log H = -\frac{1}{2} r \sigma \) for some real 3-vector \( r \), or equivalently \( H = \exp \left( -\frac{1}{2} r \sigma n \right) \). As we shall see in eq. (S.16) below, this means that \( L(H) \) is a pure Lorentz boost of rapidity \( r \) in the direction \( n \). This boost manifestly does not invert space or time, thus \( L(U) \) is proper.

Likewise, \( U \) is unitary and unimodular, thus \( U \in SU(2) \) and defines a pure rotation of space. Indeed, any \( U \in SU(2) \) can be written as \( U = \exp \left( -i \frac{\theta}{2} n' \sigma \right) \) for some angle \( \theta \) and some axis \( n' \), and according to eq. (S.16) below \( L(U) \) is indeed a pure space rotation by angle \( \theta \) around axis \( n' \). Again, this rotation is proper — it does not invert space or time. Thus, \( L(H) \) and \( L(U) \) are both proper Lorentz transforms, hence their product \( L(M) \) must also be proper. (Proof: \( \det(L(M)) = \det(L(H)) \times \det(L(U)) = +1. \) \( \text{Q.E.D.} \)

**Problem 2(c):**

\[
\sigma \lambda L_\mu^\lambda (M_2M_1) = (M_2M_1)\sigma_\mu^\lambda (M_2M_1)^\dagger = M_2 \left( M_1 \sigma_\mu^\lambda (M_1)^\dagger = \sigma_\nu^\lambda L_\nu^\mu (M_1) \right) M_2^\dagger = \left( M_2 \sigma_\nu^\lambda M_2^\dagger \right) L_\nu^\mu (M_1) = \sigma_\lambda L_\nu^\lambda (M_2) L_\nu^\mu (M_1) \tag{S.15}
\]

and hence \( L_\mu^\lambda (M_2M_1) = L_\nu^\lambda (M_2) L_\nu^\mu (M_1) \) i.e., \( L(M_2M_1) = L(M_2)L(M_1) \). \( \text{Q.E.D.} \)

**Problem 2(d):**

For \( M = \exp \left( -i \frac{\theta}{2} n \sigma \right) = \cos \frac{\theta}{2} - i \sin \frac{\theta}{2} \sigma n \) and \( M^\dagger = M^{-1} = \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \sigma n \), \( M\sigma^0M^\dagger = 1 = \sigma^0 \), which means that \( L(M) \) is merely a rotation of the 3d space. Specifically,

\[
\sigma \cdot x' = M(x\sigma)M^\dagger = \cos^2 \frac{\theta}{2} - \frac{i}{2} \sin \theta \left[ \sigma n \cdot x \right] + \sin^2 \frac{\theta}{2} (\sigma n \cdot x \sigma)(\sigma n) = \cos^2 \frac{\theta}{2} + \sin \theta (n \cdot x) \cdot \sigma + \sin^2 \frac{\theta}{2} \left( 2(n \sigma)(n \sigma) - (x \sigma) \right) = \sigma \cdot \left( \cos \theta (x - n(nx)) + \sin \theta n \times x + n(nx) \right) \tag{S.16}
\]

thus \( x' = \cos \theta (x - n(nx)) + \sin \theta n \times x + n(nx) \),

which indeed describes a rotation through angle \( \theta \) around axis \( n \).
On the other hand, for $M = M^\dagger = \exp(-\frac{r}{2} n\sigma) = \cosh\frac{r}{2} - \sinh\frac{r}{2} n\sigma$, 

$$
M(x^\mu\sigma_\mu \equiv t - x\sigma)M^\dagger = \cosh^2\frac{r}{2}(t - x\sigma) - \frac{1}{2}\sinh r \{n\sigma,t - x\sigma\}
+ \sinh^2\frac{r}{2}(n\sigma)(t - x\sigma)(n\sigma)
= \cosh^2\frac{r}{2}(t - x\sigma) - \sinh r (tn\sigma - nx)
+ \sinh^2\frac{r}{2}(t - 2(nx)(n\sigma) + (x\sigma))
= (\cosh r t + \sinh r nx) - (\sigma n)(\sinh r t + \cosh r nx)
- \sigma \cdot (x - n(nx)),
$$

(S.17)

and therefore,

$$
t' = (\cosh r) t + (\sinh r) nx, \quad x' = n((\sinh r) t + (\cosh r) nx) + (x - n(nx)),
$$

(S.18)

which is precisely the Lorentz boost of rapidity $r$ in the direction $n$. (The rapidity $r$ is related to the usual parameters of a Lorentz boost according to $\beta = \tanh r, \gamma = \cosh r, \gamma\beta = \sinh r$. For several boosts in the same directions, the rapidities add up, $r_{\text{tot}} = r_1 + r_2 + \cdots$.)

**Problem 2 (e):**

For any Lie algebra equivalent to an angular momentum or its analytic continuation, the product of two doublets comprises a triplet and a singlet, $2 \otimes 2 = 3 \oplus 1$, or in $(j)$ notations, $(\frac{1}{2}) \otimes (\frac{1}{2}) = (1) \oplus (0)$. Furthermore, the triplet $4 = (1)$ is symmetric with respect to permutations of the two doublets while the singlet $1 = (0)$ is antisymmetric.

For two separate and independent types of angular momenta $J_+$ and $J_-$ we combine the $j_+$ quantum numbers independently of $j_-$ and the $j_-$ quantum numbers independently of $j_+$. Thus,

$$
(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2}) = (1, 1) \oplus (1, 0) \oplus (0, 1) \oplus (0, 0).
$$

(S.19)

Furthermore, the symmetric part of this product should be either symmetric with respect to both the $j_+$ and the $j_-$ indices or antisymmetric with respect to both indices, thus

$$
[(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2})]_{\text{sym}} = (1, 1) \oplus (0, 0).
$$

(S.20)

Likewise, the antisymmetric part is either symmetric with respect to the $j_+$ but antisymmetric
with respect to the $j_-$ or the other way around, thus

$$ \left[ \left( \frac{1}{2}, \frac{1}{2} \right) \otimes \left( \frac{1}{2}, \frac{1}{2} \right) \right]_{\text{antisym}} = (1, 0) \oplus (0, 1). \quad (S.21) $$

From the $SO(1,3)$ point of view, the $\left( \frac{1}{2}, \frac{1}{2} \right)$ multiplet is the Lorentz vector, hence the generic 2–index Lorentz tensor decomposes into irreducible multiplets according to eq. (S.19). Imposing symmetry conditions, we have eq. (S.20) for the symmetric 2–index tensor $T^{\mu\nu} = T^{\nu\mu}$ where the singlet $(0, 0)$ corresponds to the trace $T^\mu_\mu$ while the $(1, 1)$ irreducible multiplet is the traceless symmetric tensor.

Likewise, the antisymmetric Lorentz tensor $F^{\mu\nu} = -F^{\nu\mu}$ decomposes according to eq. (S.21). Here, the irreducible components $(1, 0)$ and $(0, 1)$ are complex but conjugate to each other; individually, they describe antisymmetric tensors subject to complex duality conditions $\frac{1}{2} \epsilon^{\kappa\lambda\mu\nu} F_{\mu\nu} = \pm i F^{\kappa\lambda}$, i.e. $E = \pm i B$.

Problem 2(f):
Without the $\gamma_\mu \Psi^\mu = 0$ constraint, the spin-vector $\Psi^\mu_a$ is the tensor product or the Dirac spinor and the Lorentz vector, thus

$$ \left[ \left( \frac{1}{2}, 0 \right) \oplus (0, \frac{1}{2}) \right] \otimes \left( \frac{1}{2} \text{ half} \right) = (1, \frac{1}{2}) \oplus (0, \frac{1}{2}) \oplus (\frac{1}{2}, 1) \oplus (\frac{1}{2}, 0). \quad (S.22) $$

The constraint removes a Dirac spinor $\gamma_\mu \Psi^\mu \Rightarrow (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, thus we are left with the $(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$ part for the Rarita–Schwinger spin-vector.

Problem 3(a):
Let us evaluate $\hat{\Phi}(x' = Lx)$ according to eq. (9) but using $p' = Lp$ as an integration variable:

$$ \hat{\Phi}(x' = Lx) = \int \frac{d^3p'}{(2\pi)^3 2E_{p'}} \left[ e^{-ip'x'} \hat{a}(p') + e^{+ip'x'} \hat{a}^\dagger(p') \right]_{\mu^0 = E_{p'}} $$

$$ = \int \frac{d^3p}{(2\pi)^3 2E_p} \left[ e^{-ipx} \hat{a}(Lp) + e^{+ipx} \hat{a}^\dagger(Lp) \right]_{\mu^0 = E_p} \quad (S.23) $$

where the second equality follows from $p'x' = px$ and $\int \frac{d^3p'}{2E_{p'}} = \int \frac{d^3p}{2E_p}$. At the same time, eq. (11)
implies
\[
\hat{\Phi}(Lx) = \hat{\mathcal{D}}(L) \hat{\Phi}(x) \hat{\mathcal{D}}^\dagger(L)
\]
\[
= \int \frac{d^3p}{(2\pi)^3 2E_p} \left[ e^{-ipx} \hat{\mathcal{D}}(L) \hat{a}(p) \hat{\mathcal{D}}^\dagger(L) + e^{ipx} \hat{\mathcal{D}}(L) \hat{a}^\dagger(p) \hat{\mathcal{D}}^\dagger(L) \right]_{p^\mu = E_p} \tag{S.24}
\]

Since eqs. (S.23) and (S.24) should agree for all \(x\), the Fourier transforms of their respective right hand sides should agree for all \(p\), thus
\[
\hat{\mathcal{D}}(L) \hat{a}(p) \hat{\mathcal{D}}^\dagger(L) = \hat{a}(Lp), \quad \hat{\mathcal{D}}(L) \hat{a}^\dagger(p) \hat{\mathcal{D}}^\dagger(L) = \hat{a}^\dagger(Lp). \tag{12}
\]

Consequently
\[
\mathcal{D}(L) |p\rangle = \hat{\mathcal{D}}(L) \left( \hat{a}^\dagger(p) |0\rangle \right) = \hat{\mathcal{D}}(L) \hat{a}^\dagger(p) \left( |0\rangle = \hat{\mathcal{D}}^\dagger(L) |0\rangle \right) \\
= \left( \hat{\mathcal{D}}(L) \hat{a}^\dagger(p) \hat{\mathcal{D}}^\dagger(L) = \hat{a}^\dagger(Lp) \right) |0\rangle \\
= |Lp\rangle, \tag{13.1}
\]
and likewise
\[
\mathcal{D}(L) |p_1, p_2\rangle = \hat{\mathcal{D}}(L) \left( \hat{a}^\dagger(p_1) \hat{a}^\dagger(p_2) |0\rangle \right) \\
= \hat{\mathcal{D}}(L) \hat{a}^\dagger(p_1) \hat{a}^\dagger(p_2) \left( |0\rangle = \hat{\mathcal{D}}^\dagger(L) |0\rangle \right) \\
= \left( \hat{\mathcal{D}}(L) \hat{a}^\dagger(p_1) \hat{\mathcal{D}}^\dagger(L) \right) \left( \hat{\mathcal{D}}(L) \hat{a}^\dagger(p_2) \hat{\mathcal{D}}^\dagger(L) \right) |0\rangle \\
= \left( \hat{a}^\dagger(Lp_1) \right) \left( \hat{a}^\dagger(Lp_2) \right) |0\rangle \\
= |Lp_1, Lp_2\rangle, \tag{13.2}
\]

etc., etc. \( Q.E.D. \)

**Problem 3(b):**
Consider two sequential Lorentz transforms, first \(L_1\) and then \(L_2\). According to eq. (15) for the
combined transform $L = L_2 L_1$, 
\[
\hat{D}(L_2 L_1) \hat{\phi}_A(x) \hat{D}^\dagger(L_2 L_1) = \sum_C M_A^C ((L_2 L_1)^{-1} = L_1^{-1} L_2^{-1}) \hat{\phi}_C(L_2 L_1 x). \tag{S.25}
\]

On the other hand, applying eq. (15) first for the $L_1$ and then again for the $L_2$, we have
\[
\hat{D}(L_2) \hat{D}(L_1) \hat{\phi}_A(x) \left( \hat{D}(L_2) \hat{D}(L_1) \right)^\dagger = \hat{D}(L_2) \left( \hat{D}(L_1) \hat{\phi}_A(x) \hat{D}^\dagger(L_1) \right) \hat{D}^\dagger(L_2)
= \hat{D}(L_2) \left( \sum_B M_A^B (L_1^{-1}) \hat{\phi}_B(L_1 x) \right) \hat{D}^\dagger(L_2)
= \sum_B M_A^B (L_1^{-1}) \left( \hat{D}(L_2) \hat{\phi}_B(L_1 x) \hat{D}^\dagger(L_2) \right)
= \sum_{B,C} M_A^B (L_1^{-1}) M_B^C (L_2^{-1}) \hat{\phi}_C(L_2 L_1 x). \tag{S.26}
\]

Clearly, consistency between eqs. (S.25) and (S.26) requires that the field representation $M_A^C(L)$ and the Fock-space representation $\hat{D}(L)$ satisfy the same group law,
\[
M_A^C(L_1^{-1} L_2^{-1}) = \sum_B M_A^B (L_1^{-1}) M_B^C (L_2^{-1}) \iff \hat{D}(L_2 L_1) = \hat{D}(L_2) \hat{D}(L_1). \tag{Q.E.D.}
\]

Problem 3(c):
Reversing our derivation of eqs. (13.1–2), we see that eqs. (12) require
\[
\hat{D}(L) \hat{a}^\dagger(p, s) \hat{D}^\dagger(L) = \sum_{s'} C_{s,s'}(L, p) \hat{a}^\dagger(Lp, s'), \tag{S.27}
\]
\[
\hat{D}(L) \hat{b}^\dagger(p, s) \hat{D}^\dagger(L) = \sum_{s'} C_{s,s'}(L, p) \hat{b}^\dagger(Lp, s'),
\]
and hence by hermitian conjugation,
\[
\hat{D}(L) \hat{a}(p, s) \hat{D}^\dagger(L) = \sum_{s'} C_{s,s'}^*(L, p) \hat{a}(Lp, s'), \tag{S.28}
\]
\[
\hat{D}(L) \hat{b}(p, s) \hat{D}^\dagger(L) = \sum_{s'} C_{s,s'}^*(L, p) \hat{b}(Lp, s').
\]
Consequently, ‘sandwiching’ both sides of eq. (9) between $\hat{D}(L)$ and $\hat{D}^\dagger(L)$ operators gives us

$$
\hat{D}(L) \hat{\phi}_A(x) \hat{D}^\dagger(L) = \int \frac{d^3p}{(2\pi)^3} \sum_s \left[ e^{-ipx} f_A(p, s) \left( \hat{D}(L) \hat{a}(p, s) \hat{D}^\dagger(L) \right) + e^{ipx} h_A(p, s) \left( \hat{D}(L) \hat{b}^\dagger(p, s) \hat{D}^\dagger(L) \right) \right]_{p^0 = E_p}.
$$

On the other hand, according to eq. (15)

$$
\hat{D}(L) \hat{\phi}_A(x) \hat{D}^\dagger(L) = \sum_B M_A^B (L^{-1}) \hat{\phi}_B(x' = Lx)
$$

$$
= \sum_B M_A^B (L^{-1}) \int \frac{d^3p'}{(2\pi)^3} 2E' \sum_{s'} \left[ e^{-ip'x'} f_B(p', s') \hat{a}(p', s') + e^{ip'x'} h_B(p', s') \hat{b}^\dagger(p', s') \right]_{p'^0 = E'_p}.
$$

(S.29)

where as in eq. (S.23) $p' = Lp$ and $x' = Lx$. Comparing eqs. (S.29) and (S.30) and identifying coefficients of similar operators, we see that consistency between eqs. (15) and (17) requires

$$
\sum_s f_A(p, s) C_{s, s'}(L, p) = \sum_B M_A^B (L^{-1}) f_B(Lp, s'),
$$

$$
\sum_s h_A(p, s) C^*_{s, s'}(L, p) = \sum_B M_A^B (L^{-1}) h_B(Lp, s').
$$

(S.31)

Finally, multiplying these equations by the $M_B^A(L)$ matrix, we arrive at eqs. (18). \textbf{Q.E.D.}

As an example, consider a massive vector field $\hat{A}^\mu(x)$ which we have (in previous exercises) written in the form (16) where $\hat{b}^\dagger(p, \lambda) = \hat{a}^\dagger(p, \lambda)$ (due to hermiticity of $\hat{A}^\mu(x)$) and $f^\mu(p, \lambda)$ plays the role of $f_A(p, s)$ (as well as $h_A^*(p, s)$). Consequently, $f^\mu(p, \lambda)$ indeed transform according to eq. (18) where $M^\mu_\nu(L) = L^\mu_\nu$, as appropriate for the vector representation of the Lorentz group, while the matrix $C_{\lambda, \lambda'}$ rotates the helicity states into each other.
Similarly the Dirac spinors $u(p, s)$ and $v(p, s)$ also transform according to eqs. (18) where $M_{ab}(L)$ is the Dirac representation of the Lorentz group while the $C_{s,s'}$ matrices acts on the 3D spinors $\xi_s$ and $\eta_s$ used for construction of the Dirac spinors $u(p, s)$ and $v(p, s)$. 