An operator acting on identical bosons can be described in terms of \(N\)-particle wave functions (the *first-quantized* formalism) or in terms of creation and annihilation operators in the Fock space (the *second-quantized* formalism). This exercise is about converting the operators from one formalism to another.

In class, we defined the creation and annihilation operators \(\hat{a}^\dagger_\alpha\) and \(\hat{a}_\alpha\) in the occupation number basis according to

\[
\hat{a}^\dagger_\alpha |\{n_\beta\}\rangle = \sqrt{n_\alpha + 1} |\{n_\beta' = n_\beta + \delta_{\alpha\beta}\}\rangle,
\]

\[
\hat{a}_\alpha |\{n_\beta\}\rangle = \sqrt{n_\alpha} |\{n_\beta' = n_\beta - \delta_{\alpha\beta}\}\rangle \quad \text{(or 0 when } n_\alpha = 0\text{)}.
\]

We also wrote the wave functions of the \(|\{n_\beta\}\rangle\) states: Let \(N = \sum_\beta n_\beta\) be the number of particles and let \(|\alpha_1, \ldots, \alpha_N\rangle = |\{n_\beta\}\rangle\); then

\[
\varphi_{\alpha_1, \ldots, \alpha_N}(x_1, \ldots, x_N) = \frac{1}{\sqrt{C_{\alpha_1, \ldots, \alpha_N}}} \sum_{\text{distinct permutations} (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_N) \text{ of } (\alpha_1, \ldots, \alpha_N)} \varphi_{\tilde{\alpha}_1}(x_1) \cdots \varphi_{\tilde{\alpha}_N}(x_N), \tag{3}
\]

where \(C_{\alpha_1, \ldots, \alpha_N}\) is the number of distinct permutations.

Our first task here is to derive the wave-function action of the creation and annihilation operators (1) and (2) using eq. (3).

(a) Consider an \(N\)-particle state \(|N, \Psi\rangle\) with a completely generic totally-symmetric wave function \(\Psi(x_1, \ldots, x_N)\). Show that the \((N-1)\)-particle state \(|(N-1), \Psi'\rangle = \hat{a}_\gamma |N, \Psi\rangle\) has wave function

\[
\Psi'(x_1, \ldots, x_{N-1}) = \sqrt{N} \int d^3x_N \varphi^*_\gamma(x_N) \Psi(x_1, \ldots, x_{N-1}, x_N). \tag{4}
\]

Hint: First verify this formula for \(\Psi\) of the form (3), and then generalize to arbitrary (but totally-symmetric) \(\Psi\) by linearity.

(b) Next, show that the \((N+1)\)-particle state \(|(N+1), \Psi''\rangle = \hat{a}^\dagger_\gamma |N, \Psi\rangle\) has wave function

\[
\Psi''(x_1, \ldots, x_{N+1}) = \frac{1}{\sqrt{N+1}} \sum_{i=1}^{N+1} \varphi_\gamma(x_i) \Psi(x_1, \ldots, x_i, \ldots, x_{N+1}). \tag{5}
\]

Hint: Use the fact that \(\hat{a}^\dagger_\gamma\) is the hermitian conjugate of \(\hat{a}_\gamma\).
Now consider a one-body operator \( \hat{A}_1 \). In the first-quantized formalism \( \hat{A}_{\text{tot}} \) acts on \( N \)-particle states according to

\[
\hat{A}_{\text{tot}}^{(1)} = \sum_{i=1}^{N} \hat{A}_1 (i^{\text{th}} \text{ particle})
\]

while in the second-quantized formalism it becomes

\[
\hat{A}_{\text{tot}}^{(2)} = \sum_{\alpha, \beta} \langle \alpha | \hat{A}_1 | \beta \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta.
\]

(c) Use eq. (4) and/or eq. (5) to verify that for any two \( N \)-particle states \( \langle N, \Psi_1 \rangle \) and \( |N, \Psi_2 \rangle \)

\[
\langle N, \Psi_1 | \hat{A}_{\text{tot}}^{(1)} | N, \Psi_2 \rangle = \langle N, \Psi_1 | \hat{A}_{\text{tot}}^{(2)} | N, \Psi_2 \rangle.
\]

Hint: Use \( \hat{A}_1 = \sum_{\alpha, \beta} |\alpha \rangle \langle \alpha| \hat{A}_1 | \beta \rangle \langle \beta| \).

Next, consider a two-body operator \( \hat{B}_2 \) which acts in the first-quantized formalism according to

\[
\hat{B}_{\text{tot}}^{(1)} = \frac{1}{2} \sum_{i \neq j} \hat{B}_2 (i^{\text{th}} \text{ and } j^{\text{th}} \text{ particles})
\]

and in the second-quantized formalism according to

\[
\hat{B}_{\text{tot}}^{(2)} = \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} (\langle \alpha | \otimes \langle \beta |) \hat{B}_2 (| \gamma \rangle \otimes | \delta \rangle) \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta.
\]

(d) Again, show that for any two \( N \)-particle states \( \langle N, \Psi_1 \rangle \) and \( |N, \Psi_2 \rangle \)

\[
\langle N, \Psi_1 | \hat{B}_{\text{tot}}^{(1)} | N, \Psi_2 \rangle = \langle N, \Psi_1 | \hat{B}_{\text{tot}}^{(2)} | N, \Psi_2 \rangle.
\]

2. Next, an exercise in bosonic commutation relations

\[
[\hat{a}_\alpha, \hat{a}_\beta] = 0, \quad [\hat{a}_\alpha^\dagger, \hat{a}_\beta^\dagger] = 0, \quad [\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha \beta}.
\]

(a) Calculate the commutators \([\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\gamma^\dagger], [\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\delta^\dagger] \) and \([\hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger, \hat{a}_\gamma^\dagger \hat{a}_\delta^\dagger] \).
(b) Consider three one-body operators $\hat{A}_1$, $\hat{B}_1$, and $\hat{C}_1$. Let us define the corresponding second-quantized operators $\hat{A}_{\text{tot}}^{(2)}$, $\hat{B}_{\text{tot}}^{(2)}$, and $\hat{C}_{\text{tot}}^{(2)}$ according to eq. (7).

Show that if $\hat{C}_1 = [\hat{A}_1, \hat{B}_1]$ then $\hat{C}_{\text{tot}}^{(2)} = [\hat{A}_{\text{tot}}^{(2)}, \hat{B}_{\text{tot}}^{(2)}]$.

(c) Next, calculate the commutator $[\hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta, \hat{a}_\mu^\dagger \hat{a}_\nu]$. 

(d) Finally, let $\hat{A}_1$ be a one-body operator, let $\hat{B}_2$ and $\hat{C}_2$ be two-body operators, and let $\hat{A}_{\text{tot}}^{(2)}$, $\hat{B}_{\text{tot}}^{(2)}$, and $\hat{C}_{\text{tot}}^{(2)}$ be the corresponding second-quantized operators according to eqs. (7) and (10).

Show that if $\hat{C}_2 = \left[ \left( \hat{A}_1(1^{\text{st}}) + \hat{A}_1(2^{\text{nd}}) \right), \hat{B}_2 \right]$ then $\hat{C}_{\text{tot}}^{(2)} = [\hat{A}_{\text{tot}}^{(2)}, \hat{B}_{\text{tot}}^{(2)}]$.

3. The rest of this homework is about coherent states of harmonic oscillators and free quantum fields. Let us start with a harmonic oscillator $\hat{H} = \hbar \omega \hat{a}^\dagger \hat{a}$.

(a) For any complex number $\xi$ we define a coherent state $|\xi\rangle \overset{\text{def}}{=} \exp(\xi \hat{a}^\dagger - \xi^* \hat{a}) |0\rangle$. Show that

\[
|\xi\rangle = e^{-|\xi|^2/2} e^{\xi \hat{a}^\dagger} |0\rangle \quad \text{and} \quad \hat{a} |\xi\rangle = \xi |\xi\rangle.
\]  

(13)

(b) Calculate the uncertainties $\Delta q$ and $\Delta p$ for a coherent state $|\xi\rangle$ and verify their minimality: $\Delta q \Delta p = \frac{1}{2} \hbar$. Also, verify $\delta n = \sqrt{n}$ where $\bar{n} \overset{\text{def}}{=} \langle \hat{n} \rangle = |\xi|^2$.

Hint: use $\hat{a} |\xi\rangle = \xi |\xi\rangle$ and $\langle \xi | \hat{a}^\dagger = \xi^* \langle \xi |$.

(c) Consider time-dependent coherent states $|\xi(t)\rangle$. Show that for $\xi(t) = \xi_0 e^{-i\omega t}$, the state $|\xi(t)\rangle$ satisfies the time-dependent Schrödinger equation $i\hbar \frac{d}{dt} |\xi(t)\rangle = \hat{H} |\xi(t)\rangle$.

(d) The coherent states are not quite orthogonal to each other. Calculate their overlap $\langle \eta |\xi\rangle$.

Now consider coherent states of multi-oscillator systems and hence quantum fields. In particular, let us focus on the creation and annihilation fields $\hat{\Psi}^\dagger(x)$ and $\hat{\Psi}(x)$ for non-relativistic spinless bosons.

(e) Generalize (a) and construct coherent states $|\Phi\rangle$ which satisfy

\[
\hat{\Psi}(x) |\Phi\rangle = \Phi(x) |\Phi\rangle
\]

for any given classical complex field $\Phi(x)$. 

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(f) Show that for any such coherent state, \( \Delta N = \sqrt{\bar{N}} \) where

\[
\bar{N} \overset{\text{def}}{=} \langle \Phi | \hat{N} | \Phi \rangle = \int dx |\Phi(x)|^2.
\] (15)

(g) Let

\[
\hat{H} = \int dx \left( \frac{\hbar^2}{2M} \nabla \hat{\Psi}^\dagger \cdot \nabla \hat{\Psi} + V(x) \hat{\Psi}^\dagger \hat{\Psi} \right)
\]

and show that for any classical field configuration \( \Phi(x, t) \) that satisfies the classical field equation

\[
i\hbar \frac{\partial}{\partial t} \Phi(x, t) = \left( -\frac{\hbar^2}{2M} \nabla^2 + V(x) \right) \Phi(x, t),
\]

the time-dependent coherent state \( |\Phi\rangle \) satisfies the true Schrödinger equation

\[
\frac{i\hbar}{\partial t} \frac{\partial}{\partial t} |\Phi\rangle = \hat{H} |\Phi\rangle.
\] (16)

(h) Finally, show that the quantum overlap \( |\langle \Phi_1 | \Phi_2 \rangle|^2 \) between two different coherent states is exponentially small for any \textit{macroscopic} difference \( \delta \Phi(x) = \Phi_1(x) - \Phi_2(x) \) between the two field configurations.