1. Consider the matrix
\[ \gamma^5 \overset{\text{def}}{=} i\gamma^0\gamma^1\gamma^2\gamma^3. \]

(a) Show that \( \gamma^5 \) anticommutes with each of the \( \gamma^\mu \) matrices, \( \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5. \)

(b) Show that \( \gamma^5 \) is hermitian and that \( (\gamma^5)^2 = 1. \)

(c) Show that \( \gamma^5 = (-i/24)\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu \) and \( \gamma^{[\kappa\gamma^\lambda\gamma^\mu\gamma^\nu]} = -i\epsilon_{\kappa\lambda\mu\nu}\gamma^5. \)

(d) Show that \( \gamma^{[\lambda\gamma^\mu\gamma^\nu]} = i\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^5. \)

(e) Show that any \( 4 \times 4 \) matrix \( \Gamma \) is a unique linear combination of the following 16 matrices: 1, \( \gamma^\mu, \gamma^{[\mu\gamma^\nu]}, \gamma^5\gamma^\mu \) and \( \gamma^5. \)

Conventions: \( \epsilon^{0123} = +1, \epsilon_{0123} = -1, \gamma^{[\mu\gamma^\nu]} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu), \gamma^{[\lambda\gamma^\mu\gamma^\nu]} = \frac{1}{6}(\gamma^\lambda\gamma^\mu\gamma^\nu - \gamma^\lambda\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\mu\gamma^\lambda\gamma^\nu + \gamma^\nu\gamma^\lambda\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\lambda), \) and ditto for the \( \gamma^{[\kappa\gamma^\lambda\gamma^\mu\gamma^\nu]} \).

2. Consider bilinear products of a Dirac field \( \Psi(x) \) and its conjugate \( \overline{\Psi}(x) \). Generally, such products have form \( \overline{\Psi}\Gamma\Psi \) where \( \Gamma \) is one of 16 matrices discussed in 1.(e); altogether, we have

\[ S = \overline{\Psi}\Psi, \quad V^\mu = \overline{\Psi}\gamma^\mu\Psi, \quad T^{\mu\nu} = \overline{\Psi}i\gamma^{[\mu\gamma^\nu]}\Psi, \quad A^\mu = \overline{\Psi}\gamma^5\gamma^\mu\Psi \text{ and } P = \overline{\Psi}i\gamma^5\Psi. \]  

(1)

(a) Show that all the bilinears (1) are Hermitian.

Hint: First, show that \( (\overline{\Psi}\Gamma\Psi)^\dagger = \overline{\Psi}\Gamma\Psi \)

(b) Show that under continuous Lorentz symmetries, the \( S \) and the \( P \) transform as scalars, the \( V^\mu \) and the \( A^\mu \) as vectors and the \( T^{\mu\nu} \) as an antisymmetric tensor.

(c) Find the transformation rules of the bilinears (1) under parity (cf. problem 2 of the previous set) and show that while \( S \) is a true scalar and \( V \) is a true (polar) vector, \( P \) is a pseudoscalar and \( A \) is an axial vector.

Next, consider the charge-conjugation properties of Dirac bilinears. To avoid operator ordering problems, take \( \Psi(x) \) and \( \Psi^\dagger(x) \) to be “classical” fermionic fields which anticommute with each other, \( \Psi_\alpha\Psi_\beta^\dagger = -\Psi^\dagger_\beta\Psi_\alpha. \)
(d) In the Weyl convention, \( \hat{C}\hat{\Psi}(x)\hat{C} = \pm \gamma^2\hat{\Psi}^*(x) \). Show that \( \hat{C}\hat{\Psi}\Gamma\hat{C} = \hat{\Psi}\Gamma^c\hat{\Psi} \) where \( \Gamma^c = \gamma^0\gamma^2\Gamma^\top\gamma^0\gamma^2 \).

(e) Calculate \( \Gamma^c \) for all 16 independent matrices \( \Gamma \) and find out which Dirac bilinears are \( \mathcal{C} \)-even and which are \( \mathcal{C} \)-odd.

3. Next, an exercise in fermionic creation and annihilation operators and their anticommutation relations,
\[
\{\hat{a}_\alpha, \hat{a}_\beta\} = \{\hat{a}_\alpha^\dagger, \hat{a}_\beta\} = 0, \quad \{\hat{a}_\alpha, \hat{a}_\beta^\dagger\} = \delta_{\alpha,\beta} .
\] (2)

(a) Calculate the commutators \([\hat{a}_\alpha^\dagger\hat{a}_\beta, \hat{a}_\gamma]\), \([\hat{a}_\alpha^\dagger\hat{a}_\beta, \hat{a}_\delta]\) and \([\hat{a}_\alpha^\dagger\hat{a}_\beta, \hat{a}_\gamma^\dagger\hat{a}_\delta]\).

(b) Consider two one-body operators \( \hat{A}_1 \) and \( \hat{B}_1 \) and let \( \hat{C}_1 \) be their commutator, \( \hat{C}_1 = [\hat{A}_1, \hat{B}_1] \). Let \( \hat{A} \) be the second-quantized forms of \( \hat{A}_{\text{tot}} \),
\[
\hat{A} = \sum_{\alpha,\beta} \langle \alpha | \hat{A}_1 | \beta \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta ,
\] (3)

and ditto for the second-quantized \( \hat{B} \) and \( \hat{C} \).

Verify that \([\hat{A}, \hat{B}] = \hat{C} \).

(c) Calculate the commutator \([\hat{a}_\mu^\dagger\hat{a}_\nu, \hat{a}_\alpha^\dagger\hat{a}_\beta^\dagger\hat{a}_\gamma\hat{a}_\delta]\).

(d) The second quantized form of a two-body additive operator
\[
\hat{B}_{\text{tot}} = \frac{1}{2} \sum_{i\neq j} \hat{B}_2(i^{\text{th}} \text{ and } j^{\text{th}} \text{ particles})
\]
acting on identical fermions is
\[
\hat{B} = \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} \langle \alpha \otimes \beta | \hat{B}_2 | \gamma \otimes \delta \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta .
\] (4)

This expression is similar to its bosonic counterpart, but note the reversed order of the annihilation operators \( \hat{a}_\delta \) and \( \hat{a}_\gamma \).

Consider a one-body operator \( \hat{A}_1 \) and two two-body operators \( \hat{B}_2 \) and \( \hat{C}_2 \). Show that if \( \hat{C}_2 = \left[ (\hat{A}_1(1^{\text{st}}) + \hat{A}_1(2^{\text{nd}})) , \hat{B}_2 \right] \), then the respective second-quantized operators in the fermionic Fock space satisfy \( \hat{C} = [\hat{A}, \hat{B}] \).
4. Finally, consider the quantum Dirac fields

\[ \hat{\Psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left( e^{-ipx} u(p, s) \hat{a}_{p,s} + e^{ipx} \bar{v}(p, s) \hat{b}_{p,s}^\dagger \right) p^\mu = E_p, \]

\[ \hat{\bar{\Psi}}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left( e^{-ipx} \bar{v}(p, s) \hat{b}_{p,s} + e^{ipx} \bar{u}(p, s) \hat{a}_{p,s}^\dagger \right) p^\mu = E_p, \] (5)

where \( \hat{a}, \hat{b}, \hat{a}^\dagger, \) and \( \hat{b}^\dagger \) are relativistically normalized fermionic annihilation and creation operators, thus

\[ \{ \hat{a}_{p,s}, \hat{a}_{p',s'}^\dagger \} = \{ \hat{b}_{p,s}, \hat{b}_{p',s'}^\dagger \} = \delta_{s,s'} \times 2E_p (2\pi)^3 \delta^{(3)}(p - p') \] (6)

while all other anticommutators vanish,

\[ \{ \hat{a} \text{ or } \hat{b}, \hat{a} \text{ or } \hat{b} \} = 0, \quad \{ \hat{a}^\dagger \text{ or } \hat{b}^\dagger, \hat{a} \text{ or } \hat{b} \} = 0, \quad \{ \hat{a}, \hat{b}^\dagger \} = \{ \hat{b}, \hat{a}^\dagger \} = 0. \] (7)

As discussed in class, the free Dirac Hamiltonian is

\[ \hat{H} = \int d^3x \bar{\Psi} (-i \gamma^\mu \cdot \nabla + m) \Psi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left( E_p \hat{a}_{p,s}^\dagger \hat{a}_{p,s} + E_p \hat{b}_{p,s}^\dagger \hat{b}_{p,s} \right) + \text{const}. \] (8)

(a) Derive Dirac field’s stress-energy tensor (use Noether theorem) and show that the net mechanical momentum is

\[ \hat{\mathbf{P}}_{\text{mech}} = \int d^3x \bar{\Psi} (-i \gamma^\mu \cdot \nabla) \Psi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left( \mathbf{p} \hat{a}_{p,s}^\dagger \hat{a}_{p,s} + \mathbf{p} \hat{b}_{p,s}^\dagger \hat{b}_{p,s} \right). \] (9)

(b) Show that the electric 4-current of the electron field is \( J^\mu(x) = -e \bar{\Psi}(x) \gamma^\mu \Psi(x) \) and that the net electric charge operator is

\[ \hat{Q} = -e \int d^3x \bar{\Psi}^\dagger(x) \Psi(x) + \text{constant} \]

\[ = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left( -e \hat{a}_{p,s}^\dagger \hat{a}_{p,s} + e \hat{b}_{p,s}^\dagger \hat{b}_{p,s} \right). \] (10)

Note: The constant term in the first line arises from the operator ordering ambiguity when the classical electron field is quantized. It’s actual value — which happens to be infinite — is determined by demanding that the vacuum state has zero electric charge.
Finally, consider the net spin of electrons and positrons,

$$\hat{S}_{\text{net}} = \int d^3x \hat{\Psi}^\dagger S \hat{\Psi}. \quad (11)$$

Expand this operator into momentum modes

$$\hat{S}_{\text{net}} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \hat{S}_p \quad (12)$$

and show that for the non-relativistic modes (|p| ≪ m)

$$\hat{S}_p = \sum_{s,s'} \xi_s^\dagger \sigma \xi_{s'} \times \left( \hat{a}_{p,s}^\dagger \hat{a}_{p,s'} + \hat{b}_{p,s}^\dagger \hat{b}_{p,s'} \right) + \mathcal{O}(|p|/m). \quad (13)$$

The relativistic modes with |p| ≳ O(m) are more complicated because of mixing between the spin and the orbital angular momentum.

Hint: Approximate $u(p, s) \approx u(0, s)$ and $v(-p, s) \approx v(0, s)$ for small |p| ≪ m, and use $\eta_s = \sigma_2 \xi_s^*$. In particle terms, eqs. (8)–(13) mean that the fermionic operator $\hat{a}_{p,s}^\dagger$ creates and $\hat{a}_{p,s}$ annihilates an electron with momentum p, energy $E_p = +\sqrt{m^2 + p^2}$, spin = $\frac{1}{2}$ and spin state $\xi_s$, and electric charge = $-e$, while operator $\hat{b}_{p,s}^\dagger$ creates and $\hat{b}_{p,s}$ annihilates a positron with exactly the same momentum, energy, spin and spin state, but electric charge = $+e$. 