In the center-of-mass frame, the partial cross-section for a two particles → two particles process \( A + B \rightarrow 1 + 2 \) is calculated in the Peskin & Schroeder textbook in eq. (4.84):

\[
\left( \frac{d\sigma}{d\Omega} \right)_{\text{c.m.}} = \frac{1}{2E_A 2E_B} \frac{|p_1|}{|v_A - v_B|} \times \frac{1}{16\pi^2 E_{\text{c.m.}}} \times |M(A + B \rightarrow 1 + 2)|^2 \tag{4.84}
\]

\[
= \frac{|p_1|}{|p_A|} \times \frac{|M|^2}{64\pi^2 E_{\text{c.m.}}} \tag{S.1}
\]

When one or both of the final-state particles have spin but its polarization is not measured in the experiment, one should sum the \(|M|^2\) over the final particles’ spins. Likewise, when the initial particles have spins but the colliding beams are un-polarized — that is, the initial particles are equally likely to have any particular spin state — one should average the \(|M|^2\) over all such spin states. Thus, for the \( e^+ + e^- \rightarrow H^0 + Z^0 \) process at hand, we have

\[
\left( \frac{d\sigma}{d\Omega} \right)_{\text{c.m.}} = \frac{|p_Z|}{|p_e|} \times \frac{1}{64\pi^2 E_{\text{c.m.}}^2} \times \left( |M|^2 \overset{\text{def}}{=} \frac{1}{2} \sum_{s(e^+)} \sum_{s(e^-)} \sum_{s(Z^0)} |M|^2 \right) \tag{S.2}
\]

and according to eq. (4) of the problem set,

\[
|M(e^+ + e^- \rightarrow H^0 + Z^0)|^2 = A \left( \frac{e^2 M_{Z^0}}{E_{\text{c.m.}}^2 - M_{Z^0}^2} \right)^2 \times \left( p_{e^+} \cdot p_{e^-} + \frac{2}{M_{Z^0}^2} (p_{Z^0} \cdot p_{e^+})(p_{Z^0} \cdot p_{e^-}) \right) \tag{4}
\]

In terms of the center-of-mass-frame energies and momenta, \( p_{e^+} = -p_{e^-} \), \( E_{e^+} = E_{e^-} = \frac{1}{2} E_{\text{c.m.}} \), we have

\[
(p_{e^+} \cdot p_{e^-}) = E_{e^+}^2 + p_{e^-}^2 \approx 2E_{e^-}^2 = \frac{1}{2} E_{\text{c.m.}}^2 \tag{S.3}
\]

Also,

\[
(p_{Z^0} \cdot p_{e^\pm}) = E_{Z^0} E_{e^\pm} - p_{Z^0} \cdot p_{e^\pm} = \frac{1}{2} E_{\text{c.m.}} (E_{Z^0} \pm |p_{Z^0}| \cos \theta) \tag{S.4}
\]

where \( \theta \) is the angle between the directions of the initial electron and the final \( Z^0 \) particle — or
equivalently, between the initial positron and the final Higgs particle — and consequently

\[(p_{Z^0} \cdot p_{e^+})(p_{Z^0} \cdot p_{e^-}) = \frac{1}{4} E_{c.m.}^2 \left(E_{Z^0}^2 - p_{Z^0}^2 \cos^2 \theta\right) = \frac{1}{4} E_{c.m.}^2 \left(E_{Z^0}^2 \sin^2 \theta + M_{Z^0}^2 \cos^2 \theta\right). \quad (S.5)\]

Combining eqs. (4), (S.3) and (S.5) together, we have

\[|\mathcal{M}(e^+ + e^- \rightarrow H^0 + Z^0)|^2 = \frac{\lambda^2 A e^4 E_{c.m.}^2 M_{Z^0}^2}{(E_{c.m.}^2 - M_{Z^0}^2)^2} \times \left(2 + \frac{p_{Z^0}^2}{M_{Z^0}^2} \sin^2 \theta\right) \quad (S.6)\]

and consequently,

\[\left(\frac{d\sigma}{d\Omega}\right)_{c.m.} = \frac{|\mathcal{M}(e^+ + e^- \rightarrow H^0 + Z^0)|}{|p_e|} \times \frac{\lambda^2 A e M_{Z^0}^2}{8(E_{c.m.}^2 - M_{Z^0}^2)^2} \times \left(2 + \frac{p_{Z^0}^2}{M_{Z^0}^2} \sin^2 \theta\right) \quad (S.7)\]

The last factor here governs the angular distribution of the $Z^0$ and Higgs particles produced in the electron-positron collision. The exact shape of this distribution depends on the collision’s energy via the $p_{Z^0}^2/M_{Z^0}^2$ coefficient, so our next task is to calculate the $p_{Z^0}^2 = p_H^2$.

The energies of the $Z^0$ particle and the Higgs particle produced in the $e^+ + e^-$ collision obey the following equations:

\[E_{Z^0} + E_H = E_{c.m.}, \]
\[p_{Z^0} + p_H = 0, \]
\[E_{Z^0}^2 - E_H^2 = (p_{Z^0}^2 + M_{Z^0}^2) - (p_H^2 + M_H^2) = M_{Z^0}^2 - M_H^2, \quad (S.8)\]
\[E_{Z^0} - E_H = \frac{E_{Z^0}^2 - E_H^2}{E_{Z^0} + E_H} = \frac{M_{Z^0}^2 - M_H^2}{E_{c.m.}}, \]

and hence

\[E_{Z^0} = \frac{E_{c.m.}^2 + M_{Z^0}^2 - M_H^2}{2E_{c.m.}}, \quad E_H = \frac{E_{c.m.}^2 + M_H^2 - M_{Z^0}^2}{2E_{c.m.}} \quad (S.9)\]

while

\[p_{Z^0}^2 = p_H^2 = \frac{E_{Z^0}^2}{E_{c.m.}^2} - M_{Z^0}^2 = E_{H}^2 - M_H^2 \]
\[= \frac{1}{4 E_{c.m.}^2} \left(E_{c.m.}^4 - 2E_{c.m.}^2(M_{Z^0}^2 + M_H^2) + (M_{Z^0}^2 - M_H^2)^2\right) \quad (S.10)\]

The highest total energies available at the LEP II accelerator were barely above the $Z^0 + $ Higgs threshold — and that’s assuming the Higgs is as light as it could possibly be in light of current
experimental data. In other words,

\[ E_{\text{c.m.}} = E_{Z_0} + E_H = M_{Z_0} + M_H + \epsilon \]  (S.11)

for a rather small \( \epsilon \) (energy above the threshold), which means that the final-state particles must have low (non-relativistic) momenta. Indeed according to eq. (S.10), in this regime \( p_{Z_0}^2 \approx 2\epsilon M_{Z_0} M_H/(M_{Z_0} + M_H) \ll M_{Z_0}^2 \), and hence according to eq. (S.7) the angular distribution of the final-state particles is approximately isotropic.

At higher collision energies — hopefully available at future accelerators — the \( Z^0 \) particle would be relativistic and hence more likely to fly away perpendicular to the \( e^+ + e^- \) collision axis than along it. Ultimately, in the extremely high energy limit we have \( p_{Z_0}^2 \approx (\frac{1}{2} E_{\text{c.m.}})^2 \gg M_{Z_0}^2 \) and the partial cross section (S.7) is proportional to \( \sin^2 \theta \). In this regime, the angular distribution of the \( Z^0 \) particles — and hence the Higgs particles as well — is similar to the angular distribution of the EM radiation of an electric dipole. The reason for this similarity is that the \( Z^0 \) and the photon are both vector particles (spin=1) with similar couplings to electrons — and the ultra-relativistic \( Z^0 \) particles are in effect approximately massless.

Finally, consider the total cross-section \( \sigma \) of the \( e^+ + e^- \rightarrow Z^0 + H \) process as a function of the collision energy. Integrating eq. (S.7) over the solid angle \( d\Omega \) and substituting eq. (S.10) for the \( |p_{Z_0}| \), we arrive at

\[ \sigma_{\text{tot}}(e^+ + e^- \rightarrow Z^0 + H) = \frac{\pi A\alpha_{EM}^2}{12} \times \frac{E_{\text{c.m.}}^2}{(E_{\text{c.m.}}^2 - M_{Z_0}^2)^2} \times \left( \beta^3 + \frac{12\beta M_{Z_0}^2}{E_{\text{c.m.}}^2} \right) \]  (S.12)

where

\[ \beta \equiv \frac{|p_{Z_0}|}{\frac{1}{2} E_{\text{c.m.}}} = \sqrt{1 - \left( \frac{M_{Z_0} + M_H}{E_{\text{c.m.}}} \right)^2} \sqrt{1 - \left( \frac{M_{Z_0} - M_H}{E_{\text{c.m.}}} \right)^2} . \]  (S.13)

Near the energy threshold (S.11), the cross-section

\[ \sigma_{\text{tot}}(e^+ + e^- \rightarrow Z^0 + H) \approx \frac{2\pi \sqrt{2} A\alpha_{EM}^2 M_{Z_0}^{5/2} \epsilon^{1/2}}{M_N^{3/2}(M_H + M_{Z_0})^{3/2}(M_H + 2M_{Z_0})^2} \]  (S.14)

raises with energy as \( \sqrt{\epsilon} \), but not too far above the threshold it reaches its maximal value and
starts decreasing. At very high energies, we have

\[
\sigma(e^+ + e^- \rightarrow Z^0 + H) \approx \frac{\pi A_\text{EM}^2}{12E_{\text{c.m.}}^2} \tag{S.15}
\]

and the cross-section decreases with the energy as \(1/E_{\text{c.m.}}^2\).

For generic energies, eq. (S.12) is too complicated to study analytically. Instead, let me simply show you the numeric plot:

\[
\sigma(e^+ + e^- \rightarrow Z^0 + H^0)
\]

On this plot, \(E_{\text{c.m.}}\) is in GeV, the cross-section \(\sigma\) is in picobarns (1 pb = 10^{-36} \text{ cm}^2) and the Higgs mass is assumed to be \(M_H = 110 \text{ GeV}\), which is a bit heavier than the \(M_{Z^0} = 91.4 \text{ GeV}\).