Problem 3(a):
The difference between a circle and a straight line is that on a circle the path of a particle going from point $x_0$ to point $x'$ does not need to be ‘straight’ but may wrap around the whole circle one or more times. Indeed, let us compare a particle moving on a circle according to $x(t)$ (modulo $2\pi R$) with a particle moving on an infinite line according to $y(t)$. If the two particles have exactly the same velocities at all times,

$$\frac{dx}{dt} = \frac{dy}{dt}$$  \hspace{1cm} (S.1)

and similar initial positions $x_0 = y_0$ (according to some coordinate systems) at time $t = 0$, then after time $T$ one generally has

$$y(T) = x(T) + 2\pi R \times n$$ \hspace{1cm} (S.2)

for some integer $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ because the $x(y)$ path may wrap around the circle $n$ times while the $y(t)$ path may not wrap. For example, the two paths depicted below have same (constant) velocities and begin at $y_0 = x_0$ but end at $y(T) = x(T) + 2\pi R \times 2$:

It is easy to see that the paths $x(t)$ (modulo $2\pi R$) and $y(t)$ (modulo nothing) are in one-to-one correspondence with each other, provided we restrict the initial point $y_0$ of the particle on
the infinite line to a particular interval of length $L = 2\pi R$, say $0 \leq y_0 < 2\pi R$. Consequently, in the path integral for the particle on the circle

$$x(t=T) = x'(\mod L) \quad \int_{x(t=0) = x_0 (\mod L)}^{x(t=0) = x_0 (\mod L)} \mathcal{D}'[x(t) (\mod L)] = \sum_{n=-\infty}^{+\infty} \int_{y(t=0) = x_0}^{y(t=0) = x_0} \mathcal{D}'[y(t)].$$ (S.3)

Furthermore, in the absence of potential energy, the circle path $x(t) (\mod L)$ and the corresponding infinite line path $y(t)$ have equal actions

$$S[x(t) (\mod L)] = S[y(t)] = \int_0^T dt \left[ \frac{M}{2} \dot{x}^2 = \frac{M}{2} \dot{y}^2 \right],$$ (S.4)

and therefore

$$U_{\text{circle}}(x'; x_0) = \int_{x(t=0) = x_0 (\mod L)}^{x(t=0) = x_0 (\mod L)} \mathcal{D}'[x(t) (\mod L)] e^{iS[x(t) (\mod L)]/\hbar}$$

$$= \sum_{n=-\infty}^{+\infty} \int_{y(t=0) = x_0}^{y(t=0) = x_0} \mathcal{D}'[y(t)] e^{iS[y(t)]/\hbar}$$

$$= \sum_{n=-\infty}^{+\infty} U_{\text{line}}(y' = x' + nL; y_0 = x_0).$$ (1)

**Q.E.D.**

**Problem 3(b):**

For a free particle living on an infinite line the evolution kernel is given by

$$U_{\text{line}}(y'; y_0) = \sqrt{\frac{M}{2\pi i\hbar T}} \times \exp \left( \frac{i}{\hbar} S_{\text{classical}} = -i \frac{M}{2\hbar T} (x' - x_0)^2 \right),$$ (3)

hence according to eq. (1), a particle on a circle has kernel

$$U_{\text{circle}}(x'; x_0) = \sqrt{\frac{M}{2\pi i\hbar T}} \times \sum_{n=-\infty}^{+\infty} \exp \left( \frac{iM}{2\hbar T} (x' - x_0 + nL)^2 \right).$$ (S.5)
To evaluate this sum, we use Poisson re-summation formula (2), which gives

\[
\sum_{n=-\infty}^{+\infty} \exp \left( \frac{iM}{2\hbar T} (x' - x_0 + nL)^2 \right) = \sum_{\ell=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dv \exp \left( \frac{iM}{2\hbar T} (x' - x_0 + \nu L)^2 \right) \times e^{2\pi \nu \ell} \cdot (S.6)
\]

Rearranging the exponential, we have

\[
\frac{iM}{2\hbar T} (x' - x_0 + \nu L)^2 + 2\pi \nu \ell = \frac{iML^2}{2\hbar T} \left( \nu + \frac{x' - x_0}{L} + \frac{2\pi \nu T}{ML^2} \right) - 2\pi \nu \ell \frac{x' - x_0}{L} - \frac{i\hbar T (2\pi \nu)^2}{ML^2},
\]

and therefore

\[
\int_{-\infty}^{+\infty} dv \exp \left( \frac{iM}{2\hbar T} (x' - x_0 + \nu L)^2 \right) \times e^{2\pi \nu \ell} = \sqrt{\frac{2\pi i\hbar T}{ML^2}} \times \exp \left( -2\pi i\ell \frac{x' - x_0}{L} - \frac{(2\pi \nu)^2 i\hbar T}{ML^2} \right) \cdot (S.8)
\]

Consequently,

\[
U_{\text{circle}}(x'; x_0) = \sqrt{\frac{M}{2\pi i\hbar T}} \times \sqrt{\frac{2\pi i\hbar T}{ML^2}} \times \sum_{\ell=-\infty}^{+\infty} \exp \left( -2\pi i\ell \frac{x' - x_0}{L} - \frac{(2\pi \nu)^2 i\hbar T}{ML^2} \right)
\]

\[
= \frac{1}{L} \sum_{\ell=-\infty}^{+\infty} e^{ip(x' - x_0)/\hbar} \times e^{-iTE/\hbar}
\]

where

\[
p = -\frac{2\pi \hbar \ell}{L} = -\frac{\hbar \ell}{R} \quad \text{and} \quad E = \frac{p^2}{2M} \cdot (S.10)
\]

Problem 3(c): This is obvious from eqs. (S.9) and (S.10).