1. Consider a non-abelian gauge theory of $SU(N) \times SU(N)$ gauge fields coupled to $N^2$ complex scalars in the $(N,N)$ multiplet of the gauge group. In $N \times N$ matrix notations, the vector fields form two independent traceless hermitian matrices $B_\mu(x) = \sum_a B^a_\mu(x) \frac{\lambda^a}{2}$ and $C_\mu(x) = \sum_a C^a_\mu(x) \frac{\lambda^a}{2}$, the scalar fields form a complex matrix $\Phi$, and the Lagrangian is

$$
L = -\frac{1}{2} \text{tr}(B_{\mu\nu}B^{\mu\nu}) - \frac{1}{2} \text{tr}(C_{\mu\nu}C^{\mu\nu}) + \text{tr}(D_\mu \Phi^\dagger D^\mu \Phi) - \frac{\alpha}{2} (\text{tr}(\Phi^\dagger \Phi))^2 - \frac{\beta}{2} \text{tr}(\Phi^\dagger \Phi \Phi^\dagger \Phi) - m^2 \text{tr}(\Phi^\dagger \Phi) \tag{1}
$$

where

$$
B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu - ig[B_\mu, B_\nu],
C_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu - ig[C_\mu, C_\nu],
D_\mu \Phi = \partial_\mu \Phi + igB_\mu \Phi - igC_\mu \Phi,
D_\mu \Phi^\dagger = \partial_\mu \Phi^\dagger - ig\Phi^\dagger B_\mu + igC_\mu \Phi^\dagger. \tag{2}
$$

For simplicity, let both $SU(N)$ factors of the gauge group have the same gauge coupling $g$.

Besides the local $SU(N) \times SU(N)$ symmetries, the Lagrangian (1) has a global $U(1)$ phase symmetry $\Phi(x) \rightarrow e^{i\theta} \Phi(x)$, but some of these symmetries become spontaneously broken for $m^2 = -\mu^2 < 0$. Specifically, for $m^2 = -\mu^2 < 0$ but $\alpha, \beta > 0$, the scalar potential has a local maximum at $\Phi = 0$ while the minima lie at

$$
\Phi = \sqrt{\frac{\mu^2}{N\alpha + \beta}} \times \text{a unitary matrix}. \tag{3}
$$

All such minima are related by $SU(N) \times SU(N) \times U(1)$ symmetries to

$$
\Phi = \sqrt{\frac{\mu^2}{N\alpha + \beta}} \times 1_{N \times N}. \tag{4}
$$
(a) Which of the continuous symmetries are spontaneously broken by \( \langle \Phi \rangle \) at such a minimum and which symmetries remain unbroken?

(b) Without re-expanding the Lagrangian around the minimum (4), use Goldstone and Higgs theorems to predict the particle spectrum of the theory. List the net numbers of massive vectors, massless vectors, massive scalars, and massless scalars, and assign them to multiplets of the unbroken symmetry group.

(c) Now let’s be more specific: Show that the vector fields
\[
A_\mu^a = \frac{1}{\sqrt{2}} (B_\mu^a + C_\mu^a)
\]
remain massless while the orthogonal combinations
\[
X_\mu^a = \frac{1}{\sqrt{2}} (B_\mu^a - C_\mu^a)
\]
become massive.

Hint: Fix the unitary gauge in which the \( \Phi(x) \) matrix is hermitian up to an overall phase, \( \Phi^\dagger(x) = \Phi(x) \times e^{-2i\theta(x)} \). Explain why this gauge condition is non-singular for \( \Phi(x) \) near the minima (3).

(d) Calculate the masses of the \( X_\mu^a \) vector fields.

(e) Show that the interactions of the massless vector fields to each other are described by the \( SU(N) \) Yang–Mills Lagrangian with gauge coupling \( g_A = g/\sqrt{2} \).

Hint: To get the effective Lagrangian for just the \( A_\mu^a(x) \) fields, freeze all the other fields: \( X_\mu^a(x) \equiv 0, \Phi(x) \equiv \langle \Phi \rangle \).

(f) Finally, shift the scalar fields \( \Phi(x) \to \langle \Phi \rangle + \delta \Phi(x) \), expand the scalar potential to second order in \( \delta \Phi \) and \( \delta \Phi^\dagger \), and calculate the masses of the physical scalar particles. Check whether these masses agree with your predictions in part (b).

Hint: Mind the unitary gauge condition for the \( \Phi(x) \).

2. The rest of this exam is about Quantum Electro-Dynamics that has charged scalar particles \( S^\pm \) besides the usual photons \( \gamma \) and electrons \( e^\pm \). The Lagrangian of this theory is
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi} (i\not{\partial} - m) \Psi + D_\mu \Phi^* D^\mu \Phi - M^2 \Phi^* \Phi = \mathcal{L}^{\text{free}} + \mathcal{L}^{\text{interactions}}. \tag{5}
\]
In the Feynman rules, the propagators and the external lines follow from the \( \mathcal{L}^{\text{free}} \) while the vertices follow from the \( \mathcal{L}^{\text{interactions}} \). Altogether, the Feynman rules for QED with both electrons and charged scalars are

Photonic propagator:
\[
A^\mu \leadsto \frac{-ig^{\mu\nu}}{q^2 + i0} \rightarrow A^\nu,
\tag{F.1}
\]
Incoming photon: \[ e_\mu(k, \lambda), \] (F.2)

Outgoing photon: \[ e^*_\mu(k, \lambda), \] (F.3)

Electron propagator: \[ \Psi(q \to \overline{\Psi}) = \frac{i}{q - m + i0}, \] (F.4)

Incoming \( e^- \) or outgoing \( e^+ \): \[ = u(p, s) \text{ or } v(p, s), \] (F.5)

Outgoing \( e^- \) or incoming \( e^+ \): \[ = \bar{u}(p, s) \text{ or } \bar{v}(p, s), \] (F.6)

Scalar propagator: \[ \Phi(q \to \Phi^*) = \frac{i}{q^2 - M^2 + i0}, \] (F.7)

Incoming \( S^- \) or outgoing \( S^+ \): \[ = 1, \] (F.8)

Outgoing \( S^- \) or incoming \( S^+ \): \[ = 1, \] (F.9)

QED vertex \( ee\gamma \): \[ = +ie\gamma^\mu, \] (F.10)

Scalar QED vertex \( SS\gamma \): \[ = +ie(q_1 + q_2)^\mu \] (F.11)

Seagull vertex \( SS\gamma\gamma \): \[ = +2ie^2 g^{\mu\nu}. \] (F.12)

Note: the dotted lines (F.7–9) for the charged scalars have arrows. Also note that in the \( S^-S^+\gamma \) vertex (F.11), the directions of momenta \( q_1 \) and \( q_2 \) must agree with the arrows of the scalar lines; otherwise, the vertex becomes \( +ie(q_1 - q_2)^\mu \) or \( +ie(-q_1 + q_2)^\mu \) or \( +ie(-q_1 - q_2)^\mu \).

(a) The QED Feynman rules (F.1–6) and (F.10) were explained in class. Explain the remaining rules (F.7–9) and (F.11–12) in terms of the Lagrangian (5). Note: don’t re-derive the Feynman rules as such, just explain why the scalar lines and vertices are as in eqs. (F.7–9) and (F.11–12).
(b) Given the Feynman rules, draw the tree diagram(s) for the scalar pair production $e^-e^+ \rightarrow S^-S^+$ and calculate the tree-level amplitude $\langle S^-, S^+ | \mathcal{M} | e^-, e^+ \rangle$.

Hint: Mind the arrow directions on the dotted lines of scalars.

(c) Average $|\mathcal{M}|^2$ over the incoming particles’ spins and calculate the partial cross-section for the scalar pair production. Compare its angular dependence with that of the $e^-e^+ \rightarrow \mu^-\mu^+$ process. Also, calculate the total cross-section $\sigma_{tot}(e^-e^+ \rightarrow S^-S^+)$ and compare its energy dependence to that of $\sigma_{tot}(e^-e^+ \rightarrow \mu^-\mu^+)$. For simplicity, neglect the electron’s mass $m$. But don’t neglect the scalar’s mass $M$.

3. Finally, consider the annihilation of the charged scalars into photons, $S^+S^- \rightarrow \gamma\gamma$.

(a) Draw and evaluate all tree diagrams contributing to the $\langle \gamma\gamma | \mathcal{M} | S^+S^- \rangle$ amplitude. Make sure the amplitude respects the Bose symmetry between the two photons.

(b) Write the tree amplitude as $\mathcal{M} = \mathcal{M}_{\mu\nu} \times e^{*\mu}_1 e^{*\nu}_2$ and verify the Ward identities

$$ k^\mu_1 \times \mathcal{M}_{\mu\nu} = k^\nu_2 \times \mathcal{M}_{\mu\nu} = 0. \quad (6) $$

Hint: If these identities seem to be broken, go back to part (a) and make sure you have not missed a diagram. If this does not help, check your signs.

(c) Sum $|\mathcal{M}|^2$ over the outgoing photon polarizations and calculate the partial cross-section of the $S^+S^- \rightarrow \gamma\gamma$ annihilation.

• For simplicity, assume $E \gg M$ and neglect the scalar mass $M$ in your calculations.

★ Extra credit if you do take $M$ into account, and do it right. But beware: the kinematics is much messier for $M \neq 0$, and you might need a few hours to work through the algebra.