1. First, finish the \textit{last homework} — do problem 3 about the Feynman propagator for the massive vector field. Note: you will need some formulae from problem 2 (especially eqs. (10) and (11)) and also formulae for the scalar propagator I have derived in class.

2. Next, a reading assignment: chapter 3 of \textit{Modern Quantum Mechanics} by J. J. Sakurai, sections 1, 2, 3, second half of section 5 (representations of the rotation operators), and section 10. The other sections 4, 6, 7, 8, and 9 are not relevant to the present class material. The main focus of this assignment are the relations between the rotations and the angular momenta $\hat{J}^{x,y,z}$.

PS: If you have read the Sakurai’s book before but it has been a while, please read it again.

3. The rest of this homework is about the Lorentz group and its representations.

The continuous Lorentz group $SO^+(3,1)$ has 6 generators $\hat{J}^{\mu\nu} = -\hat{J}^{\nu\mu}$ satisfying

$$\left[\hat{J}^{\alpha\beta}, \hat{J}^{\mu\nu}\right] = ig^{\beta\mu} \hat{J}^{\alpha\nu} - ig^{\alpha\mu} \hat{J}^{\beta\nu} - ig^{\beta\nu} \hat{J}^{\alpha\mu} + ig^{\alpha\nu} \hat{J}^{\beta\mu}. \quad (1)$$

In 3D terms, the generators comprise three angular momenta $\hat{J}^i = \frac{1}{2}\epsilon^{ijk} \hat{J}^{jk}$ — which generate the rotation of space — plus 3 generators $\hat{K}^i = \hat{J}^{0i} = -\hat{J}^{i0}$ of the Lorentz boosts.

(a) Show that in 3D terms, the commutation relations (1) become

$$\left[\hat{J}^i, \hat{J}^j\right] = i\epsilon^{ijk} \hat{J}^k, \quad \left[\hat{J}^i, \hat{K}^j\right] = i\epsilon^{ijk} \hat{K}^k, \quad \left[\hat{K}^i, \hat{K}^j\right] = -i\epsilon^{ijk} \hat{J}^k. \quad (2)$$

Lorentz symmetry dictates commutation relations of $\hat{J}^{\mu\nu}$ with any operators comprising a Lorentz multiplet. In particular, for any Lorentz vector $\hat{V}^\mu$

$$\left[\hat{V}^\lambda, \hat{J}^{\mu\nu}\right] = ig^{\lambda\mu} \hat{V}^{\nu} - ig^{\lambda\nu} \hat{V}^{\mu}. \quad (3)$$

(b) Spell out these commutation relations in 3D terms, then use them to show that the Lorentz boost generators $\hat{K}$ do not commute with the Hamiltonian $\hat{H}$ of any relativistic theory.
Now consider the little group $G(p)$ of Lorentz transforms preserving the momentum vector $p^\mu$ of some massive particle, $p_\mu p^\mu = m^2 > 0$. For simplicity, assume the particle moves in $z$ direction with velocity $\beta$, thus $p^\mu = (E, 0, 0, p)$ for $E = \gamma m$ and $p = \beta \gamma m$.

(c) Show that there 3 independent combination of $\hat{J}^i$ and $\hat{K}^i$ preserving this momentum, namely

\[
\hat{J}^1 = \gamma \hat{J}^1 - \beta \gamma \hat{K}^2, \quad \hat{J}^2 = \gamma \hat{J}^2 + \beta \gamma \hat{K}^1, \quad \text{and} \quad \hat{J}^3.
\]

(d) Show that these 3 combinations commute with each other similar to the ordinary angular momenta.

For a massless particle with $p^\mu = (E, 0, 0, E)$ the little group is generated by $\hat{J}^3$, $\hat{I}^1 = \hat{J}^1 - \hat{K}^2$, and $\hat{I}^2 = \hat{J}^2 + \hat{K}^1$ which satisfy

\[
[\hat{J}^3, \hat{I}^1] = +i\hat{I}^2, \quad [\hat{J}^3, \hat{I}^2] = +i\hat{I}^1, \quad [\hat{I}^1, \hat{I}^2] = 0.
\]

As discussed in class, the finite unitary multiplets of this group are singlets made of helicity $\lambda$ satisfying $\hat{J}^3 |\lambda\rangle = \lambda |\lambda\rangle$ and $\hat{I}^{1,2} |\lambda\rangle = 0$.

(e) Show that in 4D terms the state $|p, \lambda\rangle$ of a massless particle satisfies

\[
\epsilon_{\alpha\beta\gamma\delta} \hat{J}^\beta \hat{P}^\delta |p, \lambda\rangle = -2\lambda \hat{P}_\alpha |p, \lambda\rangle.
\]

Use this formula to show that continuous Lorentz transforms do not change helicities of massless particles.

4. While particle states belong to infinite but unitary multiplets of the Lorentz group, the quantum fields form finite but non-unitary multiplets. In this problem we shall classify all such multiplets of the $SO^+(3, 1)$ group, or rather of its double cover $Spin(3, 1) \cong SL(2, \mathbb{C})$.

(a) Let’s re-organize the $\hat{J}$ and $\hat{K}$ generators of the continuous Lorentz group into two non-hermitian 3-vectors

\[
\hat{J}_+ = \frac{1}{2}(\hat{J} + i\hat{K}) \quad \text{and} \quad \hat{J}_- = \frac{1}{2}(\hat{J} - i\hat{K}) = \hat{J}_+^\dagger.
\]

Show that the two 3-vectors commute with each other, $[\hat{J}_+^k, \hat{J}_-^\ell] = 0$, while the components of each 3-vector satisfy angular momentum commutation relations, $[\hat{J}_+^k, \hat{J}_+^\ell] = i\epsilon^{k\ell m} \hat{J}_+^m$ and $[\hat{J}_-^k, \hat{J}_-^\ell] = i\epsilon^{k\ell m} \hat{J}_-^m$. 


By themselves, the $3 \hat{J}_k^+$ generate a symmetry group similar to rotations of a 3D space, but since the $\hat{J}_k^+$ are non-hermitian, the finite irreducible multiplets of this symmetry are non-unitary analytic continuations (to complex “angles”) of the ordinary angular momentum multiplets $(j)$ of spin $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$. Likewise, the finite irreducible multiplets of the symmetry group generated by the $\hat{J}_k^-$ are analytic continuations of the spin-$j$ multiplets of angular momentum. Moreover, the two symmetry groups commute with each other, so the finite irreducible multiplets of the net Lorentz symmetry are tensor products $(j_+ \otimes j_-)$ of the $\hat{J}_+$ and $\hat{J}_-$ multiplets. In other words, distinct finite irreducible multiplets of the Lorentz symmetry may be labeled by two integer or half-integer ‘spins’ $j_+$ and $j_-$, while the states within such a multiplet are $|j_+, j_-, m_+, m_-\rangle$ for $m_+ = -j_+, \ldots, +j_+$ and $m_- = -j_-, \ldots, +j_-$. The simplest non-trivial Lorentz multiplet $\mathbf{2}$ has $j_+ = \frac{1}{2}$ while $j_- = 0$. In this two-component multiplet $\hat{J}_+ = \frac{1}{2} \sigma$ while $\hat{J}_- = 0$, or in terms of $\hat{J}$ and $\hat{K}$, $\hat{J} = \frac{1}{2} \sigma$ while $\hat{K} = -\frac{i}{2} \sigma$. Consequently, the finite Lorentz transforms in this multiplet are represented by $2 \times 2$ matrices of the form

$$M = \exp \left( -ia \cdot \hat{J} - ib \cdot \hat{K} \right) = \exp \left( \frac{1}{2} (-ia + b) \cdot \sigma \right). \quad (8)$$

for some real 3-vectors $a$ and $b$. Note that such matrices always have unit determinant, $\det(M) = 1$, but there are no other general restrictions: a generic $M$ is complex, non-unitary, non-hermitian, etc., etc. The group of such $2 \times 2$ complex matrices is called $SL(2, \mathbb{C})$.

The Lorenz group has another in-equivalent two-component multiplet $\bar{\mathbf{2}}$ with $j_+ = 0$ and $j_- = \frac{1}{2}$. In this multiplet $\hat{J}$ acts as $\frac{1}{2} \sigma$ but $\hat{K}$ acts as $+\frac{i}{2} \sigma$, hence a finite Lorentz transform with the same parameters $a$ and $b$ as in eq. (8) is represented by a different $2 \times 2$ matrix

$$\bar{M} = \exp \left( \frac{1}{2} (-ia - b) \cdot \sigma \right). \quad (9)$$

Generally this matrix is in-equivalent to $M$ but rather equivalent to the complex conjugate

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* The $a$ vector parametrizes a rotation of 3D space while the $b$ vector parametrizes a Lorentz boost. A general continuous Lorentz transform involves both.
of $M$,

$$\overline{M} = \left( M^\dagger \right)^{-1} = \sigma_2 M^* \sigma_2.$$  \hspace{1cm} (10)

(b) Prove this relation for any $a$ and $b$. Hint: prove and use $\sigma_2 \sigma^* \sigma_2 = -\sigma$.

For pure rotations of 3D space, $M$ is unitary and $\overline{M} = M$. For pure Lorentz boosts, $M$ is hermitian and $\overline{M} = M^{-1}$. We shall prove both statements later in this exercise.

Later in class we shall study in great detail the Dirac spinor fields that form a reducible $2 + \bar{2}$ multiplet. There are also Weyl spinor fields that form irreducible $2$ or $\bar{2}$ multiplets. There will be future homeworks about those spinors, but for now let’s consider the other Lorentz multiplets.

In the ordinary Spin(3) = $SU(2)$ group, one can construct a multiplet of any spin $j$ from a symmetric tensor product of $2j$ doublets. This procedure gives us an object $\Phi_{\alpha_1,\ldots,\alpha_{2j}}$ with $2j$ spinor indices $\alpha_1,\ldots,\alpha_{2j} = 1,2$ that’s totally symmetric under permutation of those indices and transforms under an $SU(2)$ symmetry $U$ as

$$\Phi_{\alpha_1,\ldots,\alpha_{2j}} \rightarrow U_{\alpha_1}^{\beta_1}U_{\alpha_2}^{\beta_2}\ldots U_{\alpha_{2j}}^{\beta_{2j}} \Phi_{\beta_1,\beta_2,\ldots,\beta_{2j}}.$$  \hspace{1cm} (11)

For integer $j$, such objects are equivalent to tensors of the $SO(3)$; for example, for $j = 2$, $\Phi_{\alpha\beta} \equiv \Phi_{\beta\alpha}$ is equivalent to an $SO(3)$ vector $\vec{\Phi}$.

In the Lorentz group Spin(3,1) we have a similar situation — any multiplet can be constructed by tensoring together a bunch of two-component spinors of the $SL(2,\mathbb{C})$. But unlike the $SU(2)$, the $SL(2,\mathbb{C})$ has two different spinors $\Phi \not\sim \bar{\Phi}$ transforming under different rules. Notationally, we shall distinguish them by different index types: the undotted Greek indices belong to spinor that transform according to $M \in SL(2,\mathbb{C})$ while the dotted Greek indices belong to spinors that transform according to $M^*$ (which is equivalent to $\overline{M}$),

$$\Phi_{\alpha} \rightarrow M^{\alpha}_{\beta} \Phi_{\beta} \not\sim \Phi_{\dot{\gamma}} \rightarrow M^{*\dot{\gamma}}_{\dot{\delta}} \Phi_{\dot{\delta}}.$$  \hspace{1cm} (12)

Combining such spinors to make a multiplet with ‘spins’ $j_+$ and $j_-$, we make an object $\Phi_{\alpha_1,\ldots,\alpha_{(2j_+)};\gamma_1,\ldots,\gamma_{(2j_-)}}$ with $2j_+$ un-dotted indices and $2j_-$ dotted indices. $\Phi_{\ldots}$ is totally
symmetric under permutations of the un-dotted indices with each other or dotted indices with each other, but there is no symmetry between indices of different types. Under an SL(2, C) symmetry $M$, the un-dotted indices transform according to $M$ while the dotted indices transform according to $M^*$, thus

$$\Phi_{\alpha_1, \ldots, \alpha_{(2j_+)}, \dot{\gamma}_1, \ldots, \dot{\gamma}_{(2j_-)}} \rightarrow M_{\alpha_1} \cdots M_{\alpha_{(2j_+)}}^\beta \cdot M_{\alpha_{(2j_-)}}^{\star \delta} \cdots \cdot M_{\dot{\gamma}_1}^{\beta_{(2j_+)}} \cdots \cdot M_{\dot{\gamma}_{(2j_-)}}^{\delta_{(2j_-)}} \cdot \Phi_{\beta_1, \ldots, \beta_{(2j_+)}, \dot{\delta}_1, \ldots, \dot{\delta}_{(2j_-)}}. \quad (13)$$

Of particular importance among such multi-spinors is the bi-spinor $V_{\alpha \dot{\gamma}}$ with $j_+ = j_- = \frac{1}{2}$ — it is equivalent to the Lorentz vector $V^\mu$. The map between bi-spinors and Lorentz vectors involves four hermitian $2 \times 2$ matrices $\sigma^\mu$, where $\sigma^0$ is the unit matrix while $\sigma^1$, $\sigma^2$ and $\sigma^3$ are the Pauli matrices. In SL(2, C) terms, each $\sigma^\mu$ matrix has one dotted and one un-dotted index, thus $\sigma^\mu_{\alpha \dot{\gamma}}$. Using the $\sigma^\mu$, we may re-cast any Lorentz vector $V^\mu$ as a matrix

$$V^\mu \rightarrow V^\mu_{\mu} = V^0 - V \cdot \sigma \quad (14)$$

and hence as a $\left( \frac{1}{2}, \frac{1}{2} \right)$ bi-spinor

$$V_{\alpha \dot{\gamma}} = \left( V^\mu_{\mu} \sigma^\mu \right)_{\alpha \dot{\gamma}} = V^0 \delta_{\alpha \dot{\gamma}} - V \cdot \sigma_{\alpha \dot{\gamma}}. \quad (15)$$

Under an SL(2, C) symmetry, the bi-spinor transforms as

$$V_{\alpha \dot{\gamma}} \rightarrow V'_{\alpha \dot{\gamma}} = M_{\alpha}^{\beta} M_{\dot{\gamma}}^{\dot{\delta}} V_{\beta \dot{\delta}}' \quad (16)$$

or in matrix form,

$$V^\mu_{\mu} \sigma^\mu \rightarrow V'^\mu_{\mu} \sigma^\mu = M (V^\mu_{\mu} \sigma^\mu) M^\dagger. \quad (17)$$

Since the four matrices $\sigma^\mu$ form a complete basis of $2 \times 2$ matrices, eq. (17) defines a linear transform $V'_{\mu} = L_{\mu \nu} V_{\nu}$.

(c) Prove that for any SL(2, C) matrix $M$, the transform $L^\mu_{\mu}'(M)$ defined by eq. (17) is real (real $V'_{\mu}$ for real $V_{\mu}$), Lorentzian (preserves $V'_{\mu} V'^{\mu} = V_{\mu} V^{\mu}$) and orthochronous. Hint: prove and use $\det(V_{\mu} \sigma^\mu) = V_{\mu} V^{\mu}$. 

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For extra challenge, show that this transform is proper, $\det(L) = +1$.

(d) Verify that this $SL(2, \mathbb{C}) \to SO^+(3, 1)$ map respects the group law, $L(M_2M_1) = L(M_2)L(M_1)$.

(e) Verify explicitly that for a unitary $M = \exp\left(-\frac{i}{2}\theta \mathbf{n} \cdot \sigma\right)$, $L(M)$ is a rotation by angle $\theta$ around axis $\mathbf{n}$, while for an hermitian $M = \exp\left(\frac{1}{2}r \mathbf{n} \cdot \sigma\right)$, $L(M)$ is a boost of rapidity $r$ ($\beta = \tanh r$, $\gamma = \cosh r$) in the direction $\mathbf{n}$.

In general, any $(j_+, j_-)$ multiplet of the $SL(2, \mathbb{C})$ with integer net spin $j_+ + j_-$ is equivalent to some kind of a Lorentz tensor. (Here, we include the scalar and the vector among the tensors.) For example, the $(1, 1)$ multiplet is equivalent to a symmetric, traceless 2–index tensor $T^\mu{}\nu = T^\nu{}\mu$, $T^\mu{}_\mu = 0$. For $j_+ \neq j_-$ the representation is complex, but one can make a real tensor by combining two multiplets with opposite $j_+$ and $j_-$, for example the $(1, 0)$ and $(0, 1)$ multiplets are together equivalent to an antisymmetric 2–index tensor $F^\mu{}\nu = -F^\nu{}\mu$.

(f) Verify the above examples.

Hint: For any angular momentum $(j = \frac{1}{2}) \otimes (j = \frac{1}{2}) = (j = 1) \oplus (j = 0)$. 