Problem 1, antisymmetrization:

My notations follow the bracket convention for antisymmetrization: the indices inside square brackets are totally antisymmetrized, that is, summed with alternating signs over all $n!$ permutations. For example,

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} \equiv \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \quad (S.1)$$

or

$$2H_{\lambda\mu\nu} = \partial_{[\lambda} B_{\mu\nu]} \equiv \partial_{\lambda} B_{\mu\nu} - \partial_{\lambda} B_{\nu\mu} + \partial_{\mu} B_{\nu\lambda} - \partial_{\mu} B_{\lambda\nu} + \partial_{\nu} B_{\lambda\mu} - \partial_{\nu} B_{\mu\lambda} \quad (S.2)$$

Note that I some over all the index permutations even if the expression is already antisymmetric with respect to some of the indices, although in this case the extra permutations are equivalent to an overall combinatorial factor. For example, in eq. (1) for the $H_{\lambda\mu\nu}$ I have used the antisymmetry $B_{\mu\nu} = -B_{\nu\mu}$ to reduce the sum over $3! = 6$ permutations to just 3 cyclic permutations, each with a factor of 2 (that cancels the $\frac{1}{2}$ in the definition of the $H$):

$$H_{\lambda\mu\nu} = \frac{1}{2}(\partial_{\lambda} B_{\mu\nu} - \partial_{\lambda} B_{\nu\mu}) + \frac{1}{2}(\partial_{\mu} B_{\nu\lambda} - \partial_{\mu} B_{\lambda\nu}) + \frac{1}{2}(\partial_{\nu} B_{\lambda\mu} - \partial_{\nu} B_{\mu\lambda})$$

$$= \partial_{\lambda} B_{\mu\nu} + \partial_{\mu} B_{\nu\lambda} + \partial_{\nu} B_{\lambda\mu}. \quad (S.3)$$

Likewise, in the Jacobi identity (2), the total antisymmetry of $H_{\lambda\mu\nu}$ reduces the summation of $\partial_{[\kappa} H_{\lambda\mu\nu]}$ from $4! = 24$ terms to just 4 distinct terms.

Problem 2(a):

In the antisymmetrized-indices notations, $H_{\lambda\mu\nu} = \frac{1}{2}\partial_{[\lambda} B_{\mu\nu]}$ (cf. eq. (1)), hence

$$\partial_{[\kappa} H_{\lambda\mu\nu]} = \frac{1}{2}\partial_{[\kappa} \partial_{\lambda} B_{\mu\nu]}. \quad (S.4)$$

But the spacetime derivatives $\partial_{\kappa}$ and $\partial_{\lambda}$ commute with each other, so antisymmetrizing their indices results in total cancellation: $\partial_{[\kappa} \partial_{\lambda]}$ (whatever) = 0 and hence $\partial_{[\kappa} \partial_{\lambda} B_{\mu\nu]} = 0$. 

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Problem 1(b):
Given the Lagrangian (3) as a function of $B_{\mu\nu}$ fields and their derivatives, we have

$$\frac{\partial L(B,\partial B)}{\partial B_{\mu\nu}} = 0 \quad (S.5)$$

while

$$\frac{\partial L(B,\partial B)}{\partial (\partial_{\lambda}B_{\mu\nu})} = \frac{1}{6} H^{\alpha\beta\gamma} \times \frac{\partial H_{\alpha\beta\gamma}}{\partial (\partial_{\lambda}B_{\mu\nu})}$$

$$= \frac{1}{6} H^{\alpha\beta\gamma} \times \frac{1}{2} \delta_{\alpha}^{[\lambda} \delta_{\beta]}^{\mu} \delta_{\gamma}]$$

$$= \frac{1}{2} H_{\lambda\mu\nu}. \quad (S.6)$$

Consequently, the Euler–Lagrange field equations

$$\partial_{\lambda} \left( \frac{\partial L(B,\partial B)}{\partial (\partial_{\lambda}B_{\mu\nu})} \right) - \frac{\partial L(B,\partial B)}{\partial B_{\mu\nu}} = 0 \quad (S.7)$$

for the $B$ fields become

$$\partial_{\lambda} H_{\lambda\mu\nu} = 0. \quad (S.8)$$

Problem 1(c):
In the antisymmetrized-indices notations, the gauge transform (4) looks like

$$B'_{\mu\nu}(x) = B_{\mu\nu}(x) + \partial_{[\mu} \Lambda_{\nu]}(x). \quad (S.9)$$

Consequently,

$$H'_{\lambda\mu\nu}(x) = \frac{1}{2} \partial_{[\lambda} B'_{\mu\nu]}(x)$$

$$= \frac{1}{2} \partial_{[\lambda} B_{\mu\nu]}(x) + \frac{1}{2} \partial_{[\lambda} \partial_{\mu]} \Lambda_{\nu]}(x) \quad (S.10)$$

$$= H_{\lambda\mu\nu}(x) + 0,$$

where the last equality follows from $\partial_{[\lambda} \partial_{\mu]} = 0$. Thus, the tension fields $H_{\lambda\mu\nu}(x)$ — and hence the Lagrangian (3) — are invariant under the gauge transforms (4).
Problem 1(d):
Proceeding similarly to part (a), we have
\[ \partial_\lambda G_{\mu_1\mu_2\cdots\mu_{p+1}}(x) = \frac{1}{p!} \partial_\lambda \partial_{\mu_1} C_{\mu_2\cdots\mu_{p+1}}(x) = 0 \quad (S.11) \]
because \( \partial_\lambda \partial_{\mu_1} = 0 \). This regardless of any equations obeyed or not obeyed by the \( C(x) \) potentials, their very existence implies the Jacobi identity
\[ \partial_\lambda G_{\mu_1\mu_2\cdots\mu_{p+1}}(x) = 0 \quad (S.12) \]
for the tension fields \( G(x) \).

As to the equations of motion, the Lagrangian (8) has derivatives
\[ \frac{\partial \mathcal{L}(C, \partial C)}{\partial C_{t_1\cdots t_p}} = 0, \]
\[ \frac{\partial \mathcal{L}(C, \partial C)}{\partial (\partial_\lambda C_{t_1\cdots t_p})} = \frac{(-1)^p}{(p+1)!} G_{t_1\cdots t_{p+1}} \frac{\partial G_{t_1\cdots t_{p+1}}}{\partial (\partial_\lambda C_{t_1\cdots t_p})} \]
\[ = \frac{(-1)^p}{(p+1)!} G_{t_1\cdots t_{p+1}} \frac{1}{p!} \delta^\lambda_{t_1} \delta^\mu_{t_2} \delta^\nu_{t_3} \cdots \delta^\rho_{t_{p+1}} \]
\[ = \frac{(-1)^p}{p!} G_{t_1\cdots t_p}. \quad (S.13) \]
Hence, the Euler–Lagrange field equations
\[ \partial_\lambda \left( \frac{\partial \mathcal{L}(C, \partial C)}{\partial (\partial_\lambda C_{t_1\cdots t_p})} \right) - \frac{\partial \mathcal{L}(C, \partial C)}{\partial C_{t_1\cdots t_p}} = 0 \quad (S.14) \]
for the \( C_{\mu_1\cdots \mu_p}(x) \) fields become (up to an overall coefficient)
\[ \partial_\lambda G^{\lambda \mu_1\cdots \mu_p}(x) = 0. \quad (S.15) \]
Problem 1(e):
Under a gauge transformation (7), the $C$ tensor potential changes by

$$\Delta C_{\mu_1\cdots\mu_p}(x) = \frac{1}{(p-1)!} \partial_{[\mu_1} \Lambda_{\mu_2\cdots\mu_p]}(x)$$ (S.16)

for some arbitrary $(p-1)$–index antisymmetric tensor $\Lambda_{\mu_2\cdots\mu_p}(x)$. Hence the $G$ tension tensor changes by

$$\Delta G_{\mu_1\mu_2\cdots\mu_{p+1}}(x) = \frac{1}{p!} \partial_{[\mu_1} \Delta C_{\mu_2\cdots\mu_{p+1}]}(x) = \frac{1}{(p-1)!} \partial_{[\mu_1} \partial_{\mu_2} \Lambda_{\mu_3\cdots\mu_{p+1}]}(x),$$ (S.17)

which vanishes because $\partial_{[\mu_1} \partial_{\mu_2]} = 0$. Thus, the tension tensor $G$ is gauge invariant, and hence the Lagrangian (8) is also gauge invariant. \textit{Q.E.D.}

Mathematical Supplement to Problem 1: Differential Forms.
Mathematics of various antisymmetric tensor fields becomes much simpler in the language of differential forms. Students interested in string theory should master this language and then go ahead and learn as much differential geometry and topology as they can; take a class on the subject or at least read a book. A quick introduction to differential forms is available at Wikipedia at \url{http://en.wikipedia.org/wiki/Differential_form} and related web pages.

Consider a space or spacetime of dimension $D$; it can be Euclidean or Minkowski, flat or curved. A differential form of rank $p \leq D$ combines a tensor with $p$ indices and a differential suitable for integration over a manifold of dimension $p$ (a line for $p = 1$, a surface for $p = 2$, etc., etc.). For example,

$$A = A_\mu(x) \, dx^\mu, \quad B = B_{\mu\nu}(x) \, dx^\mu \, dx^\nu, \quad C = C_{\lambda\mu\nu}(x) \, dx^\lambda \, dx^\mu \, dx^\nu, \ldots$$ (S.18)

For $p = 2$ a 2–form should be integrated over an oriented surface, so the order of $dx^\mu$ and $dx^\nu$ matters; in fact they anticommute so $dx^\mu \, dx^\nu = -dx^\nu \, dx^\mu$. Likewise, the volume differential $dx^\lambda \, dx^\mu \, dx^\nu$ is totally antisymmetric with respect to permutation of indices $\lambda\mu\nu$. Consequently, in eq. (S.18), the $B_{\mu\nu}(x)$ tensor is antisymmetric and the $C_{\lambda\mu\nu}(x)$ tensor is totally antisymmetric in all 3 indices. And a general form of rank $p$

$$E = E_{\mu_1\mu_2\cdots\mu_p}(x) \, dx^{\mu_1} \, dx^{\mu_2} \cdots dx^{\mu_p}$$ (S.19)

involves a $p$-index totally antisymmetric tensor $E_{\mu_1\mu_2\cdots\mu_p}(x)$. 


The exterior derivative of a rank-\( p \) form \( E \) is a form \( dE \) of rank \( p+1 \) defined as

\[
dE = \partial_\lambda E_{\mu_1\mu_2\cdots\mu_p}(x) \, dx^\lambda \, dx^{\mu_1} \, dx^{\mu_2} \cdots dx^{\mu_p},
\]  

(S.20)

but this compact formula hides the antisymmetrization due to anticommutativity of the \( dx^\mu \) differentials. In the antisymmetric tensor form, \( J = dE \) means

\[
J_{\mu_1\cdots\mu_{p+1}}(x) = \frac{1}{p!} \partial_{\mu_1} E_{\mu_2\cdots\mu_{p+1}}(x) = \sum_{j=1}^{p+1} (-1)^{j-1} \partial_{\mu_j} E_{\mu_1\cdots\mu_{j-1}\mu_{j+1}\cdots\mu_{p+1}}(x).
\]  

(S.21)

The exterior derivative generalizes the 3D notions of gradient, curl, and divergence. Indeed, a scalar \( \phi(x) \) is a 0–form and its gradient \( \nabla \phi \) is a vector defining a 1-form \( (\nabla \phi)_\mu \, dx^\mu = d\phi \). Likewise, for a vector \( \vec{A}(x) \) and its curl \( \vec{B}(x) = \nabla \times \vec{A}(x) \) we have a 1-form \( A = A_\mu(x) dx^\mu \) and a 2-form \( B = B_{\mu\nu}(x) dx^\mu dx^\nu = dA \) where \( B_{\mu\nu} = \epsilon_{\mu\nu\lambda} B^\lambda = \partial_\mu A_\nu - \partial_\nu A_\mu \). (Note that for \( D = 3 \) an antisymmetric tensor is equivalent to a vector.) Finally, for a vector \( \vec{E}(x) \) and its divergence \( f(x) = \nabla \cdot \vec{E} \) we have an exterior derivative relation \( f = dE \) where \( E = E^\lambda(x) \epsilon_{\lambda\mu\nu} dx^\mu dx^\nu \) is a 2–form and \( f = f(x) \epsilon_{\lambda\mu\nu} dx^\mu dx^\nu \) is a 3–form.

The most important property of the exterior derivative is its nilpotency: for any differential form \( E, ddE = 0 \). This rule generalizes \( \nabla \times (\nabla \phi) = 0 \) and \( \nabla \cdot (\nabla \times \vec{A}) = 0 \). The proof is very simple: If \( E \) is a form of rank \( p \), \( J = dE \) is a form of rank \( p+1 \), and \( K = dJ \) is a form of rank \( p+2 \), then applying eq. (S.21) twice, we have

\[
K_{\lambda\mu_1\cdots\mu_p}(x) = \frac{1}{(p+1)!} \partial_{[\lambda} J_{\mu_1\cdots\mu_p]}(x)
= \frac{1}{(p+1)!p!} \partial_{[\lambda} \partial_{\mu} E_{\mu_1\cdots\mu_p]}(x)
\]

(S.22)

\[
= \frac{1}{p!} \partial_{[\lambda} \partial_{\mu} E_{\mu_1\cdots\mu_p}](x)
= 0
\]

where the last equality follow from \( \partial_{[\lambda} \partial_{\mu]} = 0 \).

The application of the differential form language to electromagnetic fields and to the antisymmetric tensor fields in this homework is completely straightforward. In electromagnetism we work in Minkowski spacetime and identify the 4–vector potential \( A_\mu(x) \) with a
1–form $A = A_\mu(x)dx^\mu$ and the tension tensor $F_{\mu\nu}(x)$ with a 2–form $F = F_{\mu\nu}(x)dx^\mu dx^\nu$. Clearly, $F$ is the exterior derivative of $A$:

$$F = dA \iff F_{\mu\nu} = \partial_\mu A_\nu = \partial_\mu A_\nu - \partial_\nu A_\mu.$$  \hspace{1cm} (S.23)

The Jacobi identity $\partial_\lambda F_{\mu\nu} = 0$ is simply $dF = 0$, which follows from $F = dA$ and $dF = ddA = 0$ by nilpotency of $d$. The gauge transform of the potentials is $A' = A + d\Lambda$ (where $\Lambda$ is a 0–form, \textit{i.e.} a scalar field), and the gauge invariance of the tension fields is simply

$$F' = dA' = dA + dd\Lambda = F + 0$$  \hspace{1cm} (S.24)

because $dd\Lambda = 0$.

Similarly, for the tensor potential $B_{\mu\nu}(x)$ and the tension tensor $H_{\lambda\mu\nu}$ in parts (a–c) of the problem we have a 2–form $B = B_{\mu\nu}(x)dx^\mu dx^\nu$ and a 3–form $H = H_{\lambda\mu\nu}(x)dx^\lambda dx^\mu dx^\nu$. Clearly, eq. (1) for the tension tensor translates to $H = dB$, which immediately gives rise to the Jacobi identity $dH = 0$ because $dB = 0$. Translating back to the tensor language, $dH = 0$ means eq. (2). And the gauge transform (4) is simply $B' = B + d\Lambda^{(1)}$ where $\Lambda^{(1)}$ is an arbitrary 1–form; the tension form $H$ is gauge invariant because $dd\Lambda^{(1)} = 0$.

Finally, in parts (e–f) of the problem, the totally-antisymmetric tensor potential $C_{\mu_1\cdots\mu_p}(x)$ with $p$ indices corresponds to a form $C$ of rank $p$ and the tension tensor (5) corresponds to a form $G = dC$ of rank $p + 1$. The Jacobi identity is $dG = 0$, which follows from $ddC = 0$. And the gauge transform (7) is $C' = C + d\Lambda^{(p-1)}$ for an arbitrary rank $p - 1$ form $\Lambda^{(p-1)}$; the tension form $G$ is gauge invariant because $dd\Lambda^{(p-1)} = 0$.

The Lagrangians and the equations of motion for the EM and tensor fields may also be written in the differential form language, but this is less convenient, so I am not doing it here.
Problem 2(a):
As discussed in class for the massless case (EM), \( \partial L / \partial (\partial_\mu A_\nu) = -F^{\mu \nu} \).

Clearly, \( \partial L / \partial (A_\nu) = +m^2 A^\nu - J^\nu \). Hence, the Euler–Lagrange field equation is

\[
-\partial_\mu \frac{\partial L}{\partial (\partial_\mu A_\nu)} + \frac{\partial L}{\partial (A_\nu)} = \partial_\mu F^{\mu \nu} + m^2 A^\nu - J^\nu = 0, \tag{S.25}
\]

or in terms of \( A^\nu \) and their explicit derivatives,

\[
\partial^2 A^\nu - \partial^\nu (\partial_\mu A^\mu) + m^2 A^\nu - J^\nu = 0. \tag{S.26}
\]

Problem 2(b):
Take the divergence \( \partial_\nu \) of the field equation (S.26); the first two terms cancel out while the rest becomes

\[
m^2 \partial_\nu A^\nu - \partial_\nu J^\nu = 0. \tag{S.27}
\]

In the massless case, this equation enforces the current conservation \( \partial_\nu J^\nu = 0 \) regardless of the 4–vector potential \( A^\nu (x) \), but there is no such constraint in the massive case at hand. Instead, eq. (S.27) simply relates the current divergence to the 4–potential divergence. In particular, if the current happens to satisfy \( \partial_\nu J^\nu \), then — and only then — eq. (S.27) requires \( \partial_\nu A^\nu = 0 \) as well. Consequently, the field equation (S.26) simplifies to \((\partial^2 + m^2)A^\nu = J^\nu\).

Q.E.D.

Problem 3(a):
In 3D notations, the Lagrangian (8) for the massive vector field is

\[
L = \frac{1}{2} (\mathbf{E}^2 - \mathbf{B}^2) + \frac{1}{2} m^2 (A_0^2 - \mathbf{A}^2) - J_0 A_0 + \mathbf{J} \cdot \mathbf{A}
= \frac{1}{2} (\dot{\mathbf{A}} - \nabla A_0)^2 + \frac{1}{2} (\nabla \times \mathbf{A})^2 + \frac{1}{2} m^2 (A_0^2 - \mathbf{A}^2) - J_0 A_0 + \mathbf{J} \cdot \mathbf{A}. \tag{S.28}
\]

Note that only the first term on the last line contains any time derivatives at all, and it does not contain the \( \dot{A}_0 \) but only the \( \dot{\mathbf{A}} \). Consequently, \( \partial L / \partial \dot{A}_0 = 0 \) and the scalar potential
$A_0(x)$ does not have a canonical conjugate field. On the other hand, the vector potential $A(x)$ does have a canonical conjugate, namely

$$\begin{align*}
\frac{\delta L}{\delta \dot{A}(x)} &= \left. \frac{\partial L}{\partial \dot{A}} \right|_x = -(-\dot{A}(x) - \nabla A_0(x)) = -E(x).
\end{align*}$$

(S.29)

**Problem 3(b):**
In terms of the Hamiltonian and Lagrangian densities, eq. (10) means

$$\mathcal{H} = -\dot{A} \cdot E - \mathcal{L}.$$  \hspace{1cm} (10')

Expressing all fields in terms $A$, $E$, and $A_0$, we have

$$\begin{align*}
\dot{A} &= -E - \nabla A_0, \\
-\dot{A} \cdot E &= E^2 + E \cdot \nabla A_0, \\
\mathcal{L} &= \frac{1}{2} (E^2 - (\nabla \times A)^2) + \frac{1}{2} m^2 (A_0^2 - A^2) - (A_0 J_0 - A \cdot J),
\end{align*}$$

(S.30)

and consequently,

$$\mathcal{H} = \frac{1}{2} E^2 + E \cdot \nabla A_0 - \frac{1}{2} m^2 A_0^2 + A_0 J_0 + \frac{1}{2} (\nabla \times A)^2 + \frac{1}{2} m^2 A^2 - A \cdot J.$$  \hspace{1cm} (S.31)

Taking the $\int d^3x$ integral of this density and integrating by parts the $E \cdot \nabla A_0$ term, we arrive at the Hamiltonian (11). **Q.E.D.**

**Problem 3(c):**
Evaluating the derivatives of $\mathcal{H}$ in eq. (6) gives us

$$\begin{align*}
\frac{\delta H}{\delta A_0(x)} &= \frac{\partial \mathcal{H}}{\partial A_0} - \nabla_i \frac{\partial \mathcal{H}}{\partial (\nabla_i A_0)} = -m^2 A_0 + J_0 - \nabla_i E^i.
\end{align*}$$

(S.32)

If the scalar field $A_0$ had a canonical conjugate $\pi_0(x,t)$, its time derivative $\partial \pi_0/\partial t$ would be given by the right hand side of eq. (S.32). But the $A_0(x,t)$ does not have a canonical conjugate, so instead of a Hamilton equation of motion we have a time-independent
constraint (12), namely
\[ m^2 A_0 = J_0 - \nabla \cdot E. \]  \hspace{1cm} (S.33)

In the massless EM case, a similar constraint gives rise to the Gauss Law \( \nabla \cdot E = J_0 \). But the massive vector field does not obey the Gauss Law; instead, eq. (S.33) gives us a formula for the scalar potential \( A_0 \) in terms of \( E \) and \( J_0 \).

However, Hamilton equations for the vector fields \( A \) and \( E \) are honest equations of motions. Specifically, evaluating the derivatives of \( \mathcal{H} \) in the first eq. (13), we find
\[ \frac{\delta H}{\delta E^i(x)} \equiv \frac{\partial H}{\partial (E^i)} - \nabla_j \frac{\partial H}{\partial (\nabla_j E^i)} = E^i + \nabla_i A_0, \]
which leads to Hamilton equation
\[ \frac{\partial}{\partial t} A(x, t) = -E(x, t) - \nabla A_0(x, t). \]  \hspace{1cm} (S.34)

Similarly, in the second eq. (13) we have
\[ \frac{\delta H}{\delta A^i(x)} \equiv \frac{\partial H}{\partial (A^i)} - \nabla_j \frac{\partial H}{\partial (\nabla_j A^i)} = m^2 A^i - J^i - \nabla_j (e^{ijk} (\nabla \times A)^k) \]
and hence Hamilton equation
\[ \frac{\partial}{\partial t} E(x, t) = m^2 A - J + \nabla \times (\nabla \times A). \]  \hspace{1cm} (S.35)

Problem 3(d):
In 3D notations, the Euler–Lagrange field equations (S.25) or (S.26) become
\[ \nabla \cdot E - m^2 A_0 = J_0, \]  \hspace{1cm} (S.36)
\[ -\dot{E} + \nabla \times B + m^2 A = J, \]  \hspace{1cm} (S.37)
where
\[ E \overset{\text{def}}{=} -\dot{A} - \nabla A_0, \]  \hspace{1cm} (S.38)
\[ \mathbf{B} \overset{\text{def}}{=} \nabla \times \mathbf{A}. \]  

Clearly, eq. (S.36) is equivalent to eq. (S.33) while eq. (S.37) is equivalent to eq. (S.35) (provided \( \mathbf{B} \) is defined as in eq. (S.39)). Finally, eq. (S.38) is equivalent to eq. (S.34), although their origins differ: In the Lagrangian formalism, eq. (S.38) is the definition of the \( \mathbf{E} \) field in terms of \( A_0, \mathbf{A} \) and their derivatives, while in the Hamiltonian formalism, \( \mathbf{E} \) is an independent conjugate field and eq. (S.34) is the dynamical equation of motion for the \( \dot{\mathbf{A}} \). 

Q.E.D.