Problem 1(a):
At equal times — or in the Schrödinger picture — the quantum scalar fields \( \hat{\Phi}_a(x) \) and \( \hat{\Pi}_a(x) \) satisfy commutation relations

\[
\begin{align*}
[\hat{\Phi}_a(x), \hat{\Phi}_b(y)] &\equiv 0, \\
[\hat{\Pi}_a(x), \hat{\Pi}_b(y)] &\equiv 0, \\
[\hat{\Phi}_a(x), \hat{\Pi}_b(y)] &= \delta_{ab} \times i\delta^{(3)}(x - y).
\end{align*}
\]
(S.1)

By Leibniz rule,

\[
\begin{align*}
[\hat{\Phi}_a(y)\hat{\Pi}_b(y), \hat{\Phi}_c(x)] &= \hat{\Phi}_a(y) \left[ \hat{\Pi}_b(y), \hat{\Phi}_c(x) \right] + \left[ \hat{\Phi}_a(y), \hat{\Phi}_c(x) \right] \hat{\Pi}_b(y) \\
&= \hat{\Phi}_a(y) \times (-i)\delta_{bc}\delta^{(3)}(y - x) + 0 \times \hat{\Pi}_b(y) \\
&= -i\delta_{bc} \hat{\Phi}_a(x) \times \delta^{(3)}(y - x)
\end{align*}
\]
(S.2)

and likewise

\[
\begin{align*}
[\hat{\Phi}_b(y)\hat{\Pi}_a(y), \hat{\Phi}_c(x)] &= -i\delta_{ac} \hat{\Phi}_b(x) \times \delta^{(3)}(y - x).
\end{align*}
\]
(S.3)

Hence, for the \( \hat{Q}_{ab} \) charge as in eq. (5),

\[
\begin{align*}
\left[ \hat{Q}_{ab}, \hat{\Phi}_c(x) \right] &= \int d^3 y \left[ \hat{\Phi}_a(y)\hat{\Pi}_b(y) - \hat{\Phi}_b(y)\hat{\Pi}_a(y) \right] \hat{\Phi}_c(x) \\
&= \int d^3 y \left[ -i\delta_{bc} \hat{\Phi}_a(x) + \delta_{ac} \hat{\Phi}_b(x) \right] \times \delta^{(3)}(y - x) \\
&= -i\delta_{bc} \hat{\Phi}_a(x) + \delta_{ac} \hat{\Phi}_b(x).
\end{align*}
\]
(S.4)

Similarly,

\[
\begin{align*}
\left[ \hat{\Phi}_a(y)\hat{\Pi}_b(y), \hat{\Pi}_c(x) \right] &= \hat{\Phi}_a(y) \left[ \hat{\Pi}_b(y), \hat{\Pi}_c(x) \right] + \left[ \hat{\Phi}_a(y), \hat{\Pi}_c(x) \right] \hat{\Pi}_b(y) \\
&= \hat{\Phi}_a(x) \times 0 + i\delta_{ac} \delta^{(3)}(x - y) \times \hat{\Pi}_b(y) \\
&= +i\delta_{ac} \hat{\Pi}_b(x) \times \delta^{(3)}(y - x)
\end{align*}
\]
(S.5)

and likewise

\[
\begin{align*}
\left[ \hat{\Phi}_b(y)\hat{\Pi}_a(y), \hat{\Pi}_c(x) \right] &= +i\delta_{bc} \hat{\Pi}_a(x) \times \delta^{(3)}(y - x),
\end{align*}
\]
(S.6)
hence

\[
\left[ \hat{Q}_{ab}, \hat{\Pi}_c(x) \right] = \int \! d^3 y \left[ \hat{\Phi}_a(y) \hat{\Pi}_b(y) - \hat{\Phi}_b(y) \hat{\Pi}_a(y), \hat{\Pi}_c(x) \right]
\]

\[= \int \! d^3 y \left( + i \delta_{ac} \hat{\Pi}_b(x) - i \delta_{bc} \hat{\Pi}_a(x) \right) \times \delta^{(3)}(y - x) \tag{S.7} \]

\[= - i \delta_{bc} \hat{\Pi}_a(x) + \delta_{ac} \hat{\Pi}_b(x). \]

\[Q.E.D.\]

**Problem 1(b):**

As we saw in class back in September, quantizing classical fields with Lagrangian (1) leads to Hamiltonian operator

\[
\hat{H} = \int \! d^3 x \left( \frac{1}{2} \sum_c \hat{\Pi}_c^2 + \frac{1}{2} \sum_c \nabla \hat{\Phi}_c^2 + \frac{m^2}{2} \sum_c \hat{\Phi}_c^2 + \frac{\lambda}{24} \left( \sum_c \hat{\Phi}_c^2 \right)^2 \right). \tag{S.8} \]

Each of the 4 terms on the big parentheses here is $SO(N)$ invariant, and that makes it commute with the $\hat{Q}_{ab}$ charges. Indeed, suppose some $N$ operators $\hat{V}_c$ — which could be $\hat{\Phi}_c(x)$, or $\hat{\Pi}_c(x)$, or whatever — satisfy commutation relations similar to eqs. (6), namely

\[
\left[ \hat{Q}_{ab}, \hat{V}_c \right] = - i \delta_{bc} \hat{V}_a + i \delta_{ac} \hat{V}_b, \tag{S.9} \]

then the $\sum_c \hat{V}_c^2$ operator commutes with the charges $\hat{Q}_{ab}$. Here is the proof:

\[
\left[ \hat{Q}_{ab}, \sum_c \hat{V}_c^2 \right] = \sum_c \left[ \hat{Q}_{ab}, \hat{V}_c^2 \right] = \sum_c \left\{ \hat{V}_c, \left[ \hat{Q}_{ab}, \hat{V}_c \right] \right\}
\]

\[= \sum_c \left\{ \hat{V}_c, \left( - i \delta_{bc} \hat{V}_a + i \delta_{ac} \hat{V}_b \right) \right\} \tag{S.10} \]

\[= - i \left\{ \hat{V}_b, \hat{V}_a \right\} + i \left\{ \hat{V}_a, \hat{V}_b \right\} \]

\[= 0. \]

In particular, letting $\hat{V}_c = \hat{\Pi}_c(x)$, or $\hat{V}_c = \hat{\Phi}_c(x)$, or $\hat{V}_c = \nabla \hat{\Phi}_c(x)$ — which also satisfy

\[
\left[ \hat{Q}_{ab}, \nabla \hat{\Phi}_c(x) \right] = \nabla \left[ \hat{Q}_{ab}, \hat{\Phi}_c(x) \right] = - i \delta_{bc} \nabla \hat{\Phi}_a(x) + i \delta_{ac} \nabla \hat{\Phi}_b(x) \tag{S.11} \]
— we immediately obtain

\[
\left[ \hat{Q}_{ab}, \sum_c \hat{\Pi}_c^2(x) \right] = 0, \quad \left[ \hat{Q}_{ab}, \sum_c \nabla \hat{\Phi}_c^2(x) \right] = 0, \quad \left[ \hat{Q}_{ab}, \sum_c \hat{\Phi}_c^2(x) \right] = 0, \quad \text{(S.12)}
\]

hence also

\[
\left[ \hat{Q}_{ab}, \left( \sum_c \hat{\Phi}_c^2(x) \right)^2 \right] = 0, \quad \text{(S.13)}
\]

and therefore \( \left[ \hat{Q}_{ab}, \hat{H} \right] = 0 \). \( \text{Q.E.D.} \)

**Problem 1(c):**

\[
\left[ \hat{Q}_{ab}, \hat{Q}_{cd} \right] = \left[ \hat{Q}_{ab}, \int d^3 x \left( \hat{\Phi}_c(x) \hat{\Pi}_d(x) - \hat{\Phi}_d(x) \hat{\Pi}_c(x) \right) \right]
\]

\[
= \int d^3 x \left[ \hat{Q}_{ab}, \left( \hat{\Phi}_c(x) \hat{\Pi}_d(x) - \hat{\Phi}_d(x) \hat{\Pi}_c(x) \right) \right]
\]

\[
= \int d^3 x \left( \hat{\Phi}_c(x) \left[ \hat{Q}_{ab}, \hat{\Pi}_d(x) \right] + \left[ \hat{Q}_{ab}, \hat{\Phi}_c(x) \right] \hat{\Pi}_d(x) - \hat{\Phi}_d(x) \left[ \hat{Q}_{ab}, \hat{\Pi}_c(x) \right] - \left[ \hat{Q}_{ab}, \hat{\Phi}_d(x) \right] \hat{\Pi}_c(x) \right)
\]

\[
= \int d^3 x \left( \hat{\Phi}_c \left( -i \delta_{bd} \hat{\Pi}_a + i \delta_{ad} \hat{\Pi}_b \right) + \left( -i \delta_{bc} \hat{\Phi}_a + i \delta_{ac} \hat{\Phi}_b \right) \hat{\Pi}_d - \hat{\Phi}_d \left( -i \delta_{bc} \hat{\Pi}_a + i \delta_{ac} \hat{\Pi}_b \right) - \left( -i \delta_{bd} \hat{\Phi}_a + i \delta_{ad} \hat{\Phi}_b \right) \hat{\Pi}_c \right) \hat{x}
\]

\[
= -i \delta_{bd} \times \int d^3 x \left( \hat{\Phi}_c \hat{\Pi}_a - \hat{\Phi}_a \hat{\Pi}_c \right) \hat{x} + i \delta_{ad} \times \int d^3 x \left( \hat{\Phi}_c \hat{\Pi}_b - \hat{\Phi}_b \hat{\Pi}_c \right) \hat{x}
\]

\[+ i \delta_{bc} \times \int d^3 x \left( \hat{\Phi}_d \hat{\Pi}_a - \hat{\Phi}_a \hat{\Pi}_d \right) \hat{x} - i \delta_{ac} \times \int d^3 x \left( \hat{\Phi}_d \hat{\Pi}_b - \hat{\Phi}_b \hat{\Pi}_d \right) \hat{x}
\]

\[
= -i \delta_{bd} \times \hat{Q}_{ca} + i \delta_{ad} \times \hat{Q}_{cb} + i \delta_{bc} \times \hat{Q}_{da} - i \delta_{ac} \times \hat{Q}_{db}
\]

\[
= -i \delta_{bd} \times \hat{Q}_{ad} + i \delta_{ac} \times \hat{Q}_{bd} + i \delta_{bc} \times \hat{Q}_{ac} - i \delta_{ad} \times \hat{Q}_{bc}. \quad \text{(S.14)}
\]

\( \text{Q.E.D.} \)
Problem 1(d):
In the Schrödinger picture,
\[
\hat{\Phi}_a(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \left( e^{+i\mathbf{p} \cdot x} \hat{a}_{p,a} + e^{-i\mathbf{p} \cdot x} \hat{a}_{p,a}^\dagger \right),
\]
\[
\hat{\Pi}_b(x) = \int \frac{d^3 p'}{(2\pi)^3} \frac{-iE_{p'}}{2E_{p'}} \left( e^{+i\mathbf{p}' \cdot x} \hat{a}_{p',b} - e^{-i\mathbf{p}' \cdot x} \hat{a}_{p',b}^\dagger \right).
\]
Using these expansions and
\[
\int d^3 x e^{\pm i\mathbf{p} \cdot x} e^{\pm i\mathbf{p}' \cdot x} = (2\pi)^3 \delta^{(3)}(\pm \mathbf{p} \pm \mathbf{p}'),
\]
we obtain
\[
\int d^3 x \hat{\Phi}_a(x) \hat{\Pi}_b(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \int d^3 p' \frac{-iE_{p'}}{2E_{p'}} \left( +\hat{a}_{p,a} \hat{a}_{p',b} \times (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{p}') \right.
\]
\[
\left. +\hat{a}_{p,a} \hat{a}_{p',b}^\dagger \times (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \right)
\]
\[
= \int \frac{d^3 p}{(2\pi)^3} \frac{-i}{4E_p} \left( \hat{a}_{p,a} \hat{a}_{-p,b} - \hat{a}_{p,a} \hat{a}_{p,b}^\dagger + \hat{a}_{p,a} \hat{a}_{p,b} - \hat{a}_{p,a} \hat{a}_{p,b}^\dagger \right)
\]
Likewise, exchanging \(a \leftrightarrow b\) everywhere and also \(\mathbf{p} \to -\mathbf{p}\) in two of the four terms, we have
\[
\int d^3 x \hat{\Phi}_b(x) \hat{\Pi}_a(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{-i}{4E_p} \left( \hat{a}_{-p,a} \hat{a}_{p,b} - \hat{a}_{p,a} \hat{a}_{p,b}^\dagger + \hat{a}_{p,a} \hat{a}_{p,b} - \hat{a}_{p,a} \hat{a}_{p,b}^\dagger \right).
\]
Consequently,
\[
\hat{Q}_{ab} \int d^3 x \left( \hat{\Phi}_a(x) \hat{\Pi}_b(x) \hat{\Phi}_b(x) \hat{\Pi}_a(x) \right)
\]
\[
= \int \frac{d^3 p}{(2\pi)^3} \frac{-i}{4E_p} \left( \hat{a}_{p,a} \hat{a}_{-p,b} - \hat{a}_{p,a} \hat{a}_{p,b}^\dagger + \hat{a}_{p,a} \hat{a}_{p,b} - \hat{a}_{p,a} \hat{a}_{p,b}^\dagger \right)
\]
\[
- \int \frac{d^3 p}{(2\pi)^3} \frac{-i}{4E_p} \left( \hat{a}_{-p,b} \hat{a}_{p,a} - \hat{a}_{p,b} \hat{a}_{p,a}^\dagger + \hat{a}_{p,b} \hat{a}_{p,a} - \hat{a}_{p,b} \hat{a}_{p,a}^\dagger \right)
\]
\[
= \int \frac{d^3 p}{(2\pi)^3} \frac{-i}{4E_p} \left( \hat{a}_{p,a} \hat{a}_{-p,b} - \hat{a}_{-p,a} \hat{a}_{p,b} - \hat{a}_{p,a} \hat{a}_{p,b}^\dagger + \hat{a}_{p,a} \hat{a}_{p,b} \right)
\]
\[
= \int \frac{d^3 p}{(2\pi)^3} \frac{-i}{4E_p} \left( \hat{a}_{p,a} \hat{a}_{p,b} + \hat{a}_{p,b} \hat{a}_{p,a} - \hat{a}_{p,b} \hat{a}_{p,a}^\dagger + \hat{a}_{p,a} \hat{a}_{p,b} \right)
\]
where on the last two lines
\[
\hat{a}_{p,a}\hat{a}_{-p,b} - \hat{a}_{-p,b}\hat{a}_{p,a} - \hat{a}_{p,a}\hat{a}_{-p,b} + \hat{a}_{-p,b}\hat{a}_{p,a} = \left[\hat{a}_{p,a}, \hat{a}_{-p,b}\right] - \left[\hat{a}_{p,a}, \hat{a}^\dagger_{-p,b}\right] = 0 - 0 = 0,
\]  
(S.20)
but
\[
\hat{a}_{p,a}\hat{a}_{p,b} + \hat{a}_{p,b}\hat{a}_{p,a} - \hat{a}_{p,a}\hat{a}_{p,b} + \hat{a}_{p,b}\hat{a}_{p,a} = \left[\hat{a}_{p,a}, \hat{a}_{p,b}\right] - 2\hat{a}^\dagger_{p,a}\hat{a}_{p,a} - \left[\hat{a}_{p,b}, \hat{a}_{p,a}\right] = 2\hat{Q} \tag{S.21}
\]
because
\[
\left[\hat{a}_{p,a}, \hat{a}_{p,b}\right] = \left[\hat{a}_{p,b}, \hat{a}_{p,a}\right] \tag{S.22}
\]
— although the two commutators here are infinite, it's the same infinity in both cases! Therefore, plugging eqs. (S.20) and (S.22) into the integrals on the last two lines of eq. (S.19), we obtain
\[
Q_{ab} = 0 + \int \frac{d^3p}{(2\pi)^3} \left(\frac{-iE_p}{4E_p}\right) \left(2\hat{a}_{p,a}\hat{a}_{p,b} - 2\hat{a}^\dagger_{p,a}\hat{a}_{p,b}\right)
\]
\[
= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(-i\hat{a}_{p,a}\hat{a}_{p,b} + i\hat{a}_{p,b}\hat{a}_{p,a}\right) \tag{8}
\]
\[
Q.E.D.
\]

Problem 1(e):
Action of the $Q_{ab}$ charges on single-particle states $|p, c\rangle$ follows from eq. (8) we have just proved:
\[
\hat{Q}_{ab} |p, c\rangle = -i\delta_{bc} |p, a\rangle + i\delta_{ac} |p, b\rangle \tag{S.23}
\]
and hence
\[
-\frac{i}{2} \Theta_{ab} \hat{Q}_{ab} |p, c\rangle = -\frac{1}{2} \Theta_{ac} |p, b\rangle + \frac{1}{2} \Theta_{ab} |p, b\rangle = +\Theta_{cd} |p, d\rangle. \tag{S.24}
\]
Applying this operator several times, we obtain
\[
\left(-\frac{i}{2} \Theta_{ab} \hat{Q}_{ab}\right)^2 |p, c\rangle = \left(-\frac{i}{2} \Theta_{ab} \hat{Q}_{ab}\right) \Theta_{cd} |p, d\rangle = \Theta_{cd} \left(-\frac{i}{2} \Theta_{ab} \hat{Q}_{ab}\right) |p, d\rangle = \Theta_{cd} \Theta_{de} |p, e\rangle, \tag{S.25}
\]
5
likewise
\[
\left(-\frac{i}{2} \Theta_{ab} \hat{Q}_{ab}\right)^3 |p, c\rangle = \Theta_{cd}\Theta_{de}\Theta_{ef} |p, f\rangle,
\] (S.26)

etc., etc., or in matrix form
\[
\left(-\frac{i}{2} \Theta_{ab} \hat{Q}_{ab}\right)^n |p, c\rangle = (\Theta^n)_{cd} |p, d\rangle.
\] (S.27)

Hence, expanding the exponential in eq. (9) to a power series, we obtain
\[
\hat{D} (R) |p, c\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{2} \Theta_{ab} \hat{Q}_{ab}\right)^n |p, c\rangle \\
= \sum_{n=0}^{\infty} \frac{1}{n!} (\Theta^n)_{cd} |p, d\rangle \\
= (\exp(\Theta))_{cd} |p, d\rangle \\
= R_{cd} |p, d\rangle.
\] (10)

Q.E.D.

Problem 1(f):
Back in homework set #2 we saw that Fock-space operators that are linear combinations of \(\hat{a}^\dagger \hat{a}\) terms act on multi-particle states as additive one-body-at-a-time operators,
\[
\hat{A} = \sum_{\alpha,\beta} A_{\alpha\beta} \hat{a}_\alpha^\dagger \hat{a}_\beta \rightarrow \text{N particles} \sum_{i=1}^{N} \hat{A}_1 (i\text{-th particle}) \quad \text{for} \quad \hat{A}_1 = \sum_{\alpha\beta} |\alpha\rangle A_{\alpha\beta} \langle \beta|.
\] (S.28)

In particular, the charges \(\hat{Q}_{ab}\) have form (8) of additive one-body-at-a-time operators, thus
\[
\hat{Q}_{ab}^{(N \text{ particles})} = \sum_{i=1}^{N} \hat{Q}_{ab}^{(1)} (i\text{-th particle})
\] (S.29)

where \(\hat{Q}_{ab}^{(1)} (i\text{-th})\) acts on the species index \(c\) of the \(i\text{-th}\) particle according to eq. (S.23). The \(\hat{Q}_{ab}^{(1)}\) operators acting on different particles commute with each other, so exponentiating the
$N$-particle hermitian operator (S.29) produces a product of unitary one-particle operators

$$\exp\left(-\frac{i}{2} \Theta_{ab} \hat{Q}_{ab}\right)^{(N \text{ particles})} = \bigotimes_{i=1}^{N} \exp\left(-\frac{i}{2} \Theta_{ab} \hat{Q}_{ab}\right)^{(1)} (i^{th} \text{ particle}). \quad (S.30)$$

Strictly speaking, this formula presumes $N$ independent particles whose quantum states do not have to be Bose-symmetric. However, when the operator (S.30) acts on a state whose wave function happens to be totally symmetric, the result is also totally symmetric because each one-particle factor $\exp\left(-\frac{i}{2} \Theta_{ab} \hat{Q}_{ab}\right)^{(1)} (i^{th})$ has exactly same form. Consequently, eq. (S.30) works as written also in the Hilbert space of $N$ identical bosons.

Therefore, applying the Fock-space operator (9) to an $N$-boson state with definite momenta and species of all particles would give us

$$\hat{D}(R) |(p_1, c_1), (p_2, c_2), \ldots, (p_N, c_N)\rangle =$$

$$= \bigotimes_{i=1}^{N} \exp\left(-\frac{i}{2} \Theta_{ab} \hat{Q}_{ab}\right)^{(1)} (i^{th}) |\langle p_i, c_i|\rangle$$

$$= \text{Symmetrized} \bigotimes_{i=1}^{N} \left( \exp\left(-\frac{i}{2} \Theta_{ab} \hat{Q}_{ab}\right)^{(1)} |p_i, c_i\rangle \right)$$

$$\langle\langle \text{in light of eq. (10) proved in part (e)} \rangle \rangle$$

$$= \text{Symmetrized} \bigotimes_{i=1}^{N} R_{c_i d_i} |p_i, d_i\rangle$$

$$= R_{c_1 d_1} R_{c_2 d_2} \cdots R_{c_N d_N} |\langle p_1, d_1|, (p_2, d_2), \ldots, (p_N, d_N)\rangle$$

(implicit sum over $d_1, d_2, \ldots d_N$). In other words, each particle is independently ‘rotated’ by the same $SO(N)$ symmetry $R$.

**Problem 1(g):**
The charged fields $\Phi(x)$ and $\phi^i(x)$ decompose into creation and annihilation operators for particles and antiparticles as

$$\hat{\Phi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left( e^{-ipx} \hat{a}_p^\dagger + e^{+ipx} \hat{b}_p^\dagger \right)^{p^0=+E_p},$$

$$\hat{\Phi}^i(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left( e^{-ipx} \hat{b}_p^\dagger + e^{+ipx} \hat{a}_p^\dagger \right)^{p^0=+E_p}. \quad (S.31)$$
Comparing this decomposition to the real fields

\[ \Phi_a(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \left( e^{-ipx} \hat{a}_{p,a} + e^{+ipx} \hat{a}_{p,a}^\dagger \right) p^0 = E_p, \quad a = 1, 2, \]  

we find that \( \Phi = (\Phi_1 + i\Phi_2)/\sqrt{2} \) and \( \Phi^\dagger = (\Phi_1 - i\Phi_2)/\sqrt{2} \) calls for

\[ \hat{a}_p = \frac{\hat{a}_{p,1} + i \hat{a}_{p,2}}{\sqrt{2}}, \quad \hat{b}_p = \frac{\hat{a}_{p,1} - i \hat{a}_{p,2}}{\sqrt{2}}, \quad \hat{a}_p^\dagger = \frac{\hat{a}_{p,1} - i \hat{a}_{p,2}^\dagger}{\sqrt{2}}, \quad \hat{b}_p^\dagger = \frac{\hat{a}_{p,1}^\dagger + i \hat{a}_{p,2}^\dagger}{\sqrt{2}}, \]  

(S.33)

and conversely,

\[ \hat{a}_{p,1} = \frac{\hat{a}_p + \hat{b}_p}{\sqrt{2}}, \quad \hat{a}_{p,2} = \frac{\hat{a}_p - \hat{b}_p}{\sqrt{2}i}, \quad \hat{a}_{p,1}^\dagger = \frac{\hat{a}_p^\dagger + \hat{b}_p^\dagger}{\sqrt{2}}, \quad \hat{a}_{p,2}^\dagger = \frac{\hat{a}_p^\dagger - \hat{b}_p^\dagger}{-\sqrt{2}i}. \]  

(S.34)

Consequently,

\[ -i \hat{a}_{p,2}^\dagger \hat{a}_{p,1} + i \hat{a}_{p,1}^\dagger \hat{a}_{p,2} = -i \frac{\hat{a}_p^\dagger - \hat{b}_p^\dagger}{\sqrt{2}i} \frac{\hat{a}_p + \hat{b}_p}{\sqrt{2}} + i \frac{\hat{a}_p + \hat{b}_p}{\sqrt{2}} \frac{\hat{a}_p^\dagger - \hat{b}_p^\dagger}{\sqrt{2}i} \]

\[ = \frac{1}{2} \left( \hat{a}_p^\dagger \hat{a}_p - \hat{b}_p^\dagger \hat{b}_p + \hat{a}_p^\dagger \hat{b}_p + \hat{b}_p^\dagger \hat{a}_p \right) \]

\[ + \frac{1}{2} \left( \hat{a}_p^\dagger \hat{a}_p + \hat{b}_p^\dagger \hat{b}_p - \hat{a}_p^\dagger \hat{b}_p - \hat{b}_p^\dagger \hat{a}_p \right) \]

\[ = \hat{a}_p^\dagger \hat{a}_p - \hat{b}_p^\dagger \hat{b}_p. \]  

(S.35)

Substituting this formula into eq. (8) for the \( \hat{Q}_{21} \) charge immediately gives us

\[ \hat{Q}_{21} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \left( -i \hat{a}_{p,2}^\dagger \hat{a}_{p,1} + i \hat{a}_{p,1}^\dagger \hat{a}_{p,2} \right) \]

\[ = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \left( \hat{a}_p^\dagger \hat{a}_p - \hat{b}_p^\dagger \hat{b}_p \right) \]

\[ \equiv \hat{N}_{\text{particles}} - \hat{N}_{\text{antiparticles}}. \]  

Q.E.D.
Problem 2(a):
Let $\Delta T^\mu{}^\nu = \partial_\lambda \mathcal{K}^{\lambda \mu \nu}$. Regardless of the specific form of the $\mathcal{K}^{\lambda \mu \nu}(\phi, \partial \phi)$ tensor, its antisymmetry with respect to its first two indices $\lambda$ and $\mu$ guarantees

$$ \partial_\mu \Delta T^\mu{}^\nu = \partial_\mu \partial_\lambda \mathcal{K}^{\lambda \mu \nu} = 0 \quad (S.36) $$

and hence

$$ \partial_\mu T^\mu{}^\nu = \partial_\mu T^\mu{}^\nu_{\text{Noether}} = (\text{hopefully}) = 0 \quad (15) $$

Furthermore,

$$ \int \! d^3 \mathbf{x} \ (\Delta T^0{}^\nu = \partial_i \mathcal{K}^i{}^{0 \nu}) = \oint \! d^2 \text{Area}_i \mathcal{K}^i{}^{0 \nu} \longrightarrow 0 \quad (S.37) $$

when the integral is taken over the infinite 3D space, hence

$$ P^\nu_{\text{net}} = \int \! d^3 \mathbf{x} \ T^0{}^\nu = \int \! d^3 \mathbf{x} \ T^0{}^\nu_{\text{Noether}}. \quad (16) $$

More generally, any Noether current $J^\mu_{\text{Noether}}$ can be re-defined by adding to it a total divergence of some antisymmetric tensor,

$$ J^\mu = J^\mu_{\text{Noether}} + \partial_\lambda \mathcal{T}^\lambda{}^\mu, \quad \mathcal{T}^\lambda{}^\mu = -\mathcal{T}^\mu{}^\lambda, \quad (S.38) $$

without spoiling the current conservation or changing the net charge,

$$ \partial_\mu J^\mu = \partial_\mu J^\mu_{\text{Noether}} = (\text{hopefully}) = 0, $$
$$ Q_{\text{net}} = \int \! d^3 \mathbf{x} \ J^0 = \int \! d^3 \mathbf{x} \ J^0_{\text{Noether}}. \quad (S.39) $$

The proof of these equations is completely similar to the above proof of eqs. (15) and (16).
Problem 2(b):
In the Noether’s formula (12) for the stress-energy tensor, \( \phi_a \) stand for the independent fields, however labeled. In the electromagnetic case, the independent fields are components of the 4–vector \( A_\lambda(x) \), hence

\[
T^{\mu\nu}_{\text{Noether}}(\text{EM}) = \frac{\partial L}{\partial (\partial_\mu A_\lambda)} \partial_\nu A_\lambda - g^{\mu\nu} L
= -F^{\mu\lambda} \partial_\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda}.
\]

(S.40)

While the second term here is clearly both gauge invariant and symmetric in \( \mu \leftrightarrow \nu \), the first term is neither.

Problem 2(c):
Clearly, one can easily restore both \( \mu \leftrightarrow \nu \) symmetry and gauge invariance of the electromagnetic stress-energy tensor by replacing \( \partial_\nu A_\lambda \) in eq. (S.40) with \( F_\nu^\lambda \), hence eq. (18). The correction amounts to

\[
\Delta T^{\mu\nu} = T^{\mu\nu}_{\text{true}} - T^{\mu\nu}_{\text{Noether}}
= -F^{\mu\lambda} \left( F_\lambda^\nu - \partial_\nu A_\lambda = -\partial_\lambda A_\nu \right)
= \partial_\lambda \left( F^{\mu\lambda} A_\nu \right) - A_\nu \left( \partial_\lambda F^{\mu\lambda} \right).
\]

(S.41)

Moreover, for the free EM fields, the last term on the right hand side vanishes by equation of motion \( \partial_\lambda F^{\mu\lambda} = -J^\mu = 0 \). Consequently,

\[
T^{\mu\nu} = T^{\mu\nu}_{\text{Noether}} + \partial_\lambda K^{\mu\nu}_{\lambda}
\text{where } K^{\mu\nu}_{\lambda} = F^{\mu\lambda} A_\nu = -K^{\mu\nu}_{\lambda},
\]

in perfect agreement with eq. (14).

Problem 2(d):
Let’s start with the Lagrangian (17). In component form,

\[
F^{i0} = -F^{0i} = E^i, \quad F^{ij} = -\epsilon^{ijk} B^k.
\]

(S.43)

Therefore, \( F^{i0} F_{i0} = F^{0i} F_{0i} = -E^i E^i \) where the minus sign comes from raising one space index. Likewise, \( F^{ij} F_{ij} = +\epsilon^{ijk} B^k \epsilon^{ij\ell} B^\ell = +2B^k B^k \) where the plus sign comes from raising
two space indices at once. Altogether,

\[ \mathcal{L} = -\frac{1}{4} \left( F^{\mu \nu} F_{\mu \nu} = F^{00} F_{00} + F^{0i} F_{0i} + F^{ij} F_{ij} \right) = \frac{1}{2} (E^2 - B^2). \]  

(S.44)

Consequently, eq. (18) for the energy density gives

\[ \mathcal{H} = T^{00} = -F^{0i} F_i^0 - \mathcal{L} = +E^2 - \frac{1}{2} (E^2 - B^2) = \frac{1}{2} (E^2 + B^2) \]  

(S.45)

in agreement with the standard electromagnetic formul\ae\ (note the \( c = 1 \), rationalized units here). Likewise, the energy flux and the momentum density are

\[ S^i = T^{i0} = T^{0i} = -F^{0j} F_j^i = -(E^j)(+\epsilon^{ijk} B^k) = +\epsilon^{ijk} E_j^k = (E \times B)^i, \]  

(S.46)

in agreement with the Poynting vector \( \mathbf{S} = E \times B \) (again, in the \( c = 1 \), rationalized units). Finally, the (3–dimensional) stress tensor is

\[ T^{ij}_{\text{EM}} = -F^{i\lambda} F^{j}_{\lambda} - g^{ij} \mathcal{L} = -F^{i0} F^j_0 - F^{ik} F^j_k + \delta^{ij} \mathcal{L} \]

\[ = -E^i E^j + \epsilon^{ik\ell} B^\ell \epsilon^{jkm} B^m + \frac{1}{2} \delta^{ij} (E^2 - B^2) \]  

\[ = -E^i E^j - B^i B^j + \delta^{ij} B^2 + \frac{1}{2} \delta^{ij} (E^2 - B^2) \]  

\[ = -E^i E^j - B^i B^j + \frac{1}{2} \delta^{ij} (E^2 + B^2). \]  

(S.47)

Problem 2(e):

In a sense, eq. (20) follows from eq. (S.41), but it is just as easy to derive it directly from Maxwell equations. Starting with eq. (18), we immediately have

\[ \partial_{\mu} T^{\mu \nu}_{\text{EM}} = -\left( \partial_{\mu} F^{\mu \lambda} \right) F^\nu_{\lambda} - F^{\mu \lambda} \left( \partial_{\mu} F^\nu_{\lambda} \right) + \frac{1}{2} F_{\kappa \lambda} \left( \partial^\nu F^{\kappa \lambda} \right). \]  

(S.48)

Using the antisymmetry \( F^{\mu \lambda} = -F^{\lambda \mu} \), we rewrite the second term on the right hand side as

\[ -F^{\mu \lambda} \partial_{\mu} F^\nu_{\lambda} = -F_{\mu \lambda} \partial^\mu F^{\nu \lambda} = +F_{\mu \lambda} \partial^\mu F^{\lambda \nu} \]

\[ \uparrow \text{(relabel } \lambda \leftrightarrow \mu) \]

\[ = -F_{\lambda \mu} \partial^\lambda F^{\nu \mu} = +F_{\mu \lambda} \partial^\lambda F^{\nu \mu} \]  

\[ = \frac{1}{2} F_{\mu \lambda} \left( \partial^\lambda F^{\nu \mu} + \partial^\mu F^{\lambda \nu} \right) \]

\[ = -\frac{1}{2} F_{\mu \lambda} \left( \partial^\nu F^{\mu \lambda} \right) \]  

(S.49)
where the last equality follows from the homogeneous Maxwell equation

$$\epsilon_{\kappa\lambda\mu\nu} \partial^\lambda F^\mu\nu = 0 \iff \partial^\lambda F^\nu\mu + \partial^\mu F^\lambda\nu + \partial^\nu F^\mu\lambda = 0. \quad (S.50)$$

Consequently, the second and the third terms on the right hand side of eq. (S.48) cancel each other and we are left with the first term only. Thus,

$$\partial_\mu T^{\mu\nu}_{\text{EM}} = -(\partial_\mu F^{\mu\lambda}) F^\nu_\lambda = -J^\lambda F^\nu_\lambda \quad (S.51)$$

where the second equality comes from the in-homogeneous Maxwell equation $\partial_\mu F^{\mu\lambda} = J^\lambda$. This proves eq. (20), and eq. (21) follows from that and the conservation (19) of the net stress-energy tensor. \textbf{Q.E.D.}

**Problem 3(a):**
As discussed in class, Euler–Lagrange equations for charged fields can be written in a manifestly covariant form as

$$D^\mu \frac{\partial L}{\partial (D_\mu \phi)} - \frac{\partial L}{\partial \phi} = 0. \quad (S.52)$$

In particularly, for $\phi = \Phi$, we have

$$\frac{\partial L}{\partial (D_\mu \Phi)} = D^\mu \Phi^*, \quad \frac{\partial L}{\partial \Phi} = -m^2 \Phi^*,$$

which gives us

$$D_\mu D^\mu \Phi^* + m^2 \Phi^* = 0. \quad (S.53)$$

Likewise, for $\phi = \Phi^*$ we have

$$\frac{\partial L}{\partial (D_\mu \Phi^*)} = D^\mu \Phi, \quad \frac{\partial L}{\partial \Phi^*} = -m^2 \Phi,$$

and therefore

$$D_\mu D^\mu \Phi + m^2 \Phi = 0. \quad (S.54)$$

As for the vector fields $A_\nu$, the Lagrangian (22) depends on $\partial_\mu A_\nu$ only through $F_{\mu\nu}$, which
gives us the usual Maxwell equation
\[ \partial_\mu F^{\mu \nu} = J^\nu \quad \text{where} \quad J^\nu \equiv -\frac{\partial L}{\partial A_\nu}. \] (S.55)

To obtain the current \( J^\nu \), we notice that the covariant derivatives of the charged fields \( \Phi \) and \( \Phi^\ast \) depend on the gauge field:
\[ \frac{\partial D_\mu \Phi}{\partial A_\nu} = iq\delta_\mu^\nu \Phi, \quad \frac{\partial D_\mu \Phi^\ast}{\partial A_\nu} = -iq\delta_\mu^\nu \Phi^\ast. \] (S.56)

Consequently,
\[ J^\nu = -\frac{\partial L}{\partial D_\nu \Phi} \times (iq\Phi) - \frac{\partial L}{\partial D_\nu \Phi^\ast} \times (-iq\Phi^\ast) \] (S.57)
\[ = -iq(\Phi D^\nu \Phi^\ast - \Phi^\ast D^\nu \Phi). \]

Note that all derivatives on the last line here are gauge-covariant, which makes the current \( J^\nu \) gauge invariant. In a non-covariant form,
\[ J^\nu = iq\Phi^\ast \partial^\nu \Phi - iq\Phi \partial^\nu \Phi^\ast - 2q^2 \Phi^\ast \Phi A^\nu. \] (S.58)

To prove the conservation of this current, we use the Leibniz rule for covariant derivatives, \( D_\nu (XY) = XD_\nu Y + YD_\nu X \). This gives us
\[ \partial_\mu (\Phi^\ast D^\mu \Phi) = D_\mu (\Phi^\ast D^\mu \Phi) = (D_\mu \Phi^\ast)(D^\mu \Phi) + \Phi^\ast (D_\mu D^\mu \Phi), \]
\[ \partial_\mu (\Phi D^\mu \Phi^\ast) = D_\mu (\Phi D^\mu \Phi^\ast) = (D_\mu \Phi)(D^\mu \Phi^\ast) + \Phi (D_\mu D^\mu \Phi^\ast), \] (S.59)

and hence in light of eq. (S.57) for the current,
\[ \partial_\nu J^\nu = -iq \left( (D_\nu \Phi)(D^\nu \Phi^\ast) + \Phi(D_\nu D^\nu \Phi^\ast) \right) + iq \left( (D_\nu \Phi^\ast)(D^\nu \Phi) + \Phi^\ast (D_\nu D^\nu \Phi) \right) \]
\[ = iq\Phi^\ast D^2 \Phi - iq\Phi D^2 \Phi^\ast \]
\[ \langle \text{by equations of motion} \rangle \]
\[ = iq\Phi^\ast (-m^2 \Phi) - iq\Phi (-m^2 \Phi^\ast) \]
\[ = 0. \] (S.60)

\textbf{Q.E.D.}
Problem 3(b):

According to the Noether theorem,

\[
T_{\text{Noether}}^{\mu\nu} = \frac{\partial L}{\partial (\partial_\mu A_\lambda)} \partial^\nu A_\lambda + \frac{\partial L}{\partial (\partial_\phi A_\lambda)} \partial^\nu A_\lambda + \frac{\partial L}{\partial (\partial_\phi \Phi)} \partial^\nu \Phi^* - g^{\mu\nu} L
\]

(S.61)

where

\[
T_{\text{Noether}}^{\mu\nu}(\text{EM}) = -F^{\mu\lambda} \partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda}
\]

(S.62)

similar to the free EM fields, and

\[
T_{\text{Noether}}^{\mu\nu}(\text{matter}) = D^\mu \Phi^*(\partial^\nu \Phi) + D^\mu \phi (D^\nu \Phi^* - \partial^\nu \Phi^*) - g^{\mu\nu} (D^\lambda \Phi \partial^\nu \Phi - m^2 \Phi \Phi^*)
\]

(S.63)

Both terms on the second line of eq. (S.61) lack \( \mu \leftrightarrow \nu \) symmetry and gauge invariance and thus need \( \partial_\lambda K^{\lambda\mu\nu} \) corrections for some \( K^{\lambda\mu\nu} = -K^{\mu\lambda\nu} \). We would like to show that the same \( K^{\lambda\mu\nu} = -F^{\mu\lambda} A^\nu \) we used to improve the free electromagnetic stress-energy tensor will now improve both the \( T_{\text{EM}}^{\mu\nu} \) and \( T_{\text{mat}}^{\mu\nu} \) at the same time!

Indeed, to improve the scalar fields’ stress-energy tensor we need

\[
\Delta T_{\text{matter}}^{\mu\nu} \equiv T_{\text{true}}^{\mu\nu}(\text{matter}) - T_{\text{Noether}}^{\mu\nu}(\text{matter})
\]

\[
= D^\mu \Phi^*(D^\nu \Phi - \partial^\nu \Phi) + D^\mu \phi (D^\nu \Phi^* - \partial^\nu \Phi^*)
\]

\[
= D^\mu \Phi^*(iq A^\nu \Phi) + D^\mu \phi (-iq A^\nu \Phi^*)
\]

\[
= -A^\nu (iq \Phi^* D^\mu \Phi - iq \Phi D^\mu \Phi^*)
\]

\[
= -A^\nu J^\mu,
\]

(S.64)

while the improvement of the EM stress-energy tensor was spelled out in eq. (S.41):

\[
\Delta T_{\text{EM}}^{\mu\nu} = -F^{\mu\lambda}(F_{\lambda}^\nu - \partial^\nu A_\lambda) = +F^{\mu\lambda} \partial_\lambda A^\nu = \partial_\lambda (-F^{\lambda\mu} A^\nu) + A^\nu J^\mu.
\]

(S.65)

Altogether, to symmetrize the whole stress-energy tensor, we need

\[
\Delta T_{\text{tot}}^{\mu\nu} \equiv T_{\text{true}}^{\mu\nu}(\text{total}) - T_{\text{Noether}}^{\mu\nu}(\text{total}) = \partial_\lambda \left( F^{\mu\lambda} A^\nu \equiv K^{\lambda\mu\nu} \right).
\]

Q.E.D.
Problem 3(c):
Because the fields $\Phi(x)$ and $\Phi^*(x)$ have opposite electric charges, their product is neutral and therefore $\partial_\mu(\Phi^*\Phi) = D_\mu(\Phi^*\Phi) = (D_\mu\Phi^*)\Phi + \Phi^*(D_\mu\Phi)$. Similarly,

$$
\partial_\mu ((D^\mu \Phi^*) (D^\nu \Phi)) = (D_\mu D^\mu \Phi^*) (D^\nu \Phi) + (D^\mu \Phi^*) (D_\mu D^\nu \Phi) = -m^2 \Phi^* (D^\nu \Phi) + (D_\mu \Phi^*) (D_\mu D^\nu \Phi + iqF^{\mu\nu} \Phi) \tag{S.66}
$$

where we have applied the field equation $(D_\mu D^\mu + m^2)\Phi^*(x) = 0$ to the first term on the right hand side and used $[D^\mu, D^\nu] = iqF^{\mu\nu}$ to expand the second term. Likewise,

$$
\partial_\mu ((D^\mu \Phi) (D^\nu \Phi^*)) = (D_\mu D^\mu \Phi) (D^\nu \Phi^*) + (D^\mu \Phi) (D_\mu D^\nu \Phi^*) = -m^2 \Phi (D^\nu \Phi^*) + (D_\mu \Phi) (D_\mu D^\nu \Phi^* - iqF^{\mu\nu} \Phi) \tag{S.67}
$$

and

$$
\partial_\mu \left[ -g^{\mu\nu} \left( D_\lambda \Phi^* D^\lambda \Phi - m^2 \Phi^* \Phi \right) \right] = -\partial^\nu \left( D_\lambda \Phi^* D^\lambda \Phi \right) + m^2 \partial^\nu (\Phi^* \Phi) = - (D^\nu D^\mu \Phi^*) (D_\mu \Phi) - (D_\mu \Phi^*) (D^\nu D^\mu \Phi) + m^2 \Phi (D^\nu \Phi^*) + m^2 \Phi^* (D^\nu \Phi) . \tag{S.68}
$$

Together, the left hand sides of eqs. (S.66), (S.67) and (S.68) comprise $\partial_\mu T^{\mu\nu}_{\text{mat}}$ — cf. eq. (26). On the other hand, combining the right hand sides of these three equations results in massive cancellation of all terms except those containing the gauge field strength tensor $F^{\mu\nu}$. Specifically,

$$
\partial_\mu T^{\mu\nu}_{\text{mat}} = (D_\mu \Phi^*) (iqF^{\mu\nu} \Phi) + (D_\mu \Phi) (-iqF^{\mu\nu} \Phi^*) = F^{\mu\nu} (iq\Phi D_\mu \Phi^* - iq\Phi^* D_\mu \Phi) \tag{S.69}
$$

In perfect agreement in eq. (21). In light of eq. (20) proved in problem 2(e), this means that the total stress-energy tensor (25) is conserved. Q.E.D.