Problem 1(a):
The ‘Dirac sandwich’

$$\bar{v} \gamma_\nu u = v^\dagger \gamma^0 \gamma_\nu u = v^\dagger \begin{pmatrix} \sigma_\nu & 0 \\ 0 & \bar{\sigma}_\nu \end{pmatrix} u$$  \hspace{1cm} (S.1)

does not mix chiralities: if $u$ and $v$ are chiral, then they should have the same chirality, both left or both right, or else $\bar{v} \gamma_\nu u = 0$. By inspection of eqs. (2), the $u$ spinor of an ultra-relativistic electron has chirality matching the electron’s helicity, left for $\lambda = -\frac{1}{2}$ and right for $\lambda = +\frac{1}{2}$. But the $v$ spinor or an ultra-relativistic positron has the opposite chirality, left for $\lambda = +\frac{1}{2}$ and right for $\lambda = -\frac{1}{2}$. Thus, to have the same chiralities of $u$ and $v$ for the sake of $\bar{v} \gamma_\nu u$, the electron and the positron must have opposite helicities. If the have the same helicities as in eq. (3), they have opposite chiralities and $\bar{v} \gamma_\nu u = 0$. \textit{Q.E.D.}

Indeed, let’s calculate the sandwich (S.1) for the spinors (2):

$$\bar{v}(e^+_L) \gamma_\nu u(e^-_L) = -2E \begin{pmatrix} 0 \\ \eta_L \end{pmatrix}^\dagger \begin{pmatrix} \sigma_\nu & 0 \\ 0 & \bar{\sigma}_\nu \end{pmatrix} \begin{pmatrix} \xi_L \\ 0 \end{pmatrix} = 0,$$  \hspace{1cm} (3)

$$\bar{v}(e^+_L) \gamma_\nu u(e^-_R) = -2E \begin{pmatrix} 0 \\ \eta_L \end{pmatrix}^\dagger \begin{pmatrix} \sigma_\nu & 0 \\ 0 & \bar{\sigma}_\nu \end{pmatrix} \begin{pmatrix} 0 \\ \xi_R \end{pmatrix} = -2E \times \eta_L^\dagger \bar{\sigma}_\nu \xi_R,$$  \hspace{1cm} (S.2)

$$\bar{v}(e^+_R) \gamma_\nu u(e^-_L) = +2E \begin{pmatrix} \eta_R \\ 0 \end{pmatrix}^\dagger \begin{pmatrix} \sigma_\nu & 0 \\ 0 & \bar{\sigma}_\nu \end{pmatrix} \begin{pmatrix} \xi_L \\ 0 \end{pmatrix} = +2E \times \eta_R^\dagger \sigma_\nu \xi_L,$$  \hspace{1cm} (S.3)

$$\bar{v}(e^+_R) \gamma_\nu u(e^-_R) = +2E \begin{pmatrix} \eta_R \\ 0 \end{pmatrix}^\dagger \begin{pmatrix} \sigma_\nu & 0 \\ 0 & \bar{\sigma}_\nu \end{pmatrix} \begin{pmatrix} 0 \\ \xi_R \end{pmatrix} = 0.$$  \hspace{1cm} (3)

Eqs. (3) have important practical consequences for electron-positron colliders. Any kind of fermion pair production — $\mu^- \mu^+$, or $\tau^- \tau^+$, or $q\bar{q}$ — which proceeds through a virtual vector particle — a photon, or $Z^0$, or even something not yet discovered — would have the $\bar{v}(e^+_\gamma) \gamma_\nu u(e^-)$ factor in the amplitude. According to eq. (3), the electron and the positron must have opposite helicities, or they would not annihilate each other and make pairs.

Now suppose we have a longitudinally polarized electron beam — say $\lambda = +\frac{1}{2}$ only — but the positron beam is un-polarized. Because of eq. (3), only the left-handed positrons would
collide with the right-handed electrons and produce pairs, while the left-handed positrons would do something else. Likewise, if we give the electron beam \( \lambda = -\frac{1}{2} \) polarization, then only the right-handed positrons would collide with our left-handed electrons and make pairs, while the left-handed positrons would do something else. Thus, as far as the pair-production is concerned, the positron beam could just as well be longitudinally polarized with \( \lambda(e^+) = -\lambda(e^-) \).

In other words, if we want to study polarization effects in fermion pair production, it’s enough to longitudinally polarize just the electron beam. We do not need to polarize the positron beam — which is much harder to do — because electrons of definite helicity would automatically select positrons of the opposite helicity.

Problem 1(b):
For ultra-relativistic muons, the \( u(\mu^-) \) and \( v(\mu^+) \) are chiral, and the chiralities behave exactly similar to the electron and positron in part (a): the Dirac sandwich \( \bar{u}(\mu^-)\gamma^\nu v(\mu^+) \) vanishes unless \( u \) and \( v \) have the same chirality and hence the \( \mu^- \) and the \( \mu^+ \) have opposite helicities. Indeed,

\[
\bar{u}(\mu^-_L)\gamma^\nu v(\mu^+_L) = -2E \begin{pmatrix} \xi_L \end{pmatrix}^\dagger \begin{pmatrix} \sigma_{\nu} & 0 \\ 0 & \sigma_{\nu} \end{pmatrix} \begin{pmatrix} 0 \\ \eta_L \end{pmatrix} = 0, \tag{4}
\]

\[
\bar{u}(\mu^-_L)\gamma^\nu v(\mu^+_R) = -2E \begin{pmatrix} \xi_L \end{pmatrix}^\dagger \begin{pmatrix} \sigma_{\nu} & 0 \\ 0 & \sigma_{\nu} \end{pmatrix} \begin{pmatrix} \eta_R \\ 0 \end{pmatrix} = +2E \times \xi_L^\dagger \sigma_{\nu} \eta_R, \tag{S.4}
\]

\[
\bar{u}(\mu^-_R)\gamma^\nu v(\mu^+_L) = -2E \begin{pmatrix} 0 \end{pmatrix}^\dagger \begin{pmatrix} \sigma_{\nu} & 0 \\ 0 & \sigma_{\nu} \end{pmatrix} \begin{pmatrix} 0 \\ \eta_L \end{pmatrix} = -2E \times \xi_R^\dagger \sigma_{\nu} \eta_L, \tag{S.5}
\]

\[
\bar{u}(\mu^-_R)\gamma^\nu v(\mu^+_R) = -2E \begin{pmatrix} 0 \end{pmatrix}^\dagger \begin{pmatrix} \sigma_{\nu} & 0 \\ 0 & \sigma_{\nu} \end{pmatrix} \begin{pmatrix} \eta_R \\ 0 \end{pmatrix} = 0. \tag{4}
\]

Eqs. (4) — and similar formulae for other fermion-antifermion pairs produced with ultra-relativistic speeds in electron-positron collisions — assure that the fermion and the antifermion always have opposite helicities. Experimentally, this means that if for some event we are able to determine the helicity of one final particle, then we may infer the second final particle’s helicity without any further experimental effort.
Problem 1(c):
The electron moves in the positive $z$ direction, so its helicity $\lambda$ is the same as its $S_z$ — the $z$ component of its spin. Hence, the $\xi$ spinors corresponding to the 2 helicities are

$$\xi(e^-_L) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \xi(e^-_R) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$ (S.6)

The positron moves in the negative $z$ direction, so its helicity is opposite from $S_z$, hence

$$\xi(e^+_L) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \xi(e^+_R) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$ (S.7)

The $\eta$ spinors in eqs. (2) for the positrons are related to $\xi$ spinors as $\eta = \sigma_2 \xi$, thus

$$\eta(e^+_L) = \begin{pmatrix} 0 \\ +i \end{pmatrix} \quad \text{and} \quad \eta(e^+_R) = \begin{pmatrix} -i \\ 0 \end{pmatrix}.$$ (S.8)

Substituting these 2–component spinors into eqs. (S.2) and (S.3), we obtain

$$\bar{v}(e^+_L) \gamma^\nu u(e^-_R) = -2E \times (0 \ -i) \bar{\sigma}^\nu \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= +2iE \times (\bar{\sigma})_{21}$$

$$= 2E \times (0, +i, +1, 0)^\nu,$$ (5)

$$\bar{v}(e^+_R) \gamma^\nu u(e^-_L) = +2E \times (+i \ 0) \sigma^\nu \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= +2iE \times (\sigma)_{12}$$

$$= 2E \times (0, -i, +1, 0)^\nu.$$ When taking the 21 and 12 matrix elements of the $\bar{\sigma}^\nu$ and $\sigma^\nu$ matrices, please remember that $\bar{\sigma}^\nu = (1, \sigma^x, \sigma^y, \sigma^z)$ while $\sigma^\nu = (1, -\sigma^x, -\sigma^y, -\sigma^z).$
Problem 1(d):
Suppose for a moment $\theta = 0$ and the $\mu^\mp$ move in the same directions as $e^\mp$. Then the muons have exactly the same spinors $u$ and $v$ as the $e^\mp$ of the same charge and helicity, hence

$$\bar{v}(\mu_L^+)\gamma^\nu u(\mu_R^-) = 2E \times (0, +i, +1, 0)^\nu \quad \text{and} \quad \bar{v}(\mu_R^+)\gamma^\nu u(\mu_L^-) = 2E \times (0, -i, +1, 0)^\nu$$

(S.9)

exactly as in eqs. (5). The $\bar{u}(\mu^-)\gamma^\nu v(\mu^+)$ Dirac sandwiches we need for the amplitude (1) follow by complex conjugation:

$$\left(\bar{v}\gamma^\nu u\right)^* = \bar{u}\gamma^\nu v = \bar{u}\gamma^\nu v,$$

(S.10)

and hence

$$\bar{u}(\mu_R^-)\gamma^\nu v(\mu_L^+) = 2E \times (0, +i, +1, 0)^{\nu*} = 2E \times (0, -i, +1, 0)^\nu,$$

$$\bar{u}(\mu_L^-)\gamma^\nu v(\mu_R^+) = 2E \times (0, -i, +1, 0)^{\nu*} = 2E \times (0, +i, +1, 0)^\nu.$$  

(S.11)

Eqs. (S.11) apply for $\theta = 0$. For other muon directions, we may simply rotate the 4–vectors (S.11) through angle $\theta$ in the $xz$ plane, thus

$$\bar{u}(\mu_R^-)\gamma^\nu v(\mu_L^+) = 2E \times (0, -i \cos \theta, +1, +i \sin \theta)^\nu,$$

$$\bar{u}(\mu_L^-)\gamma^\nu v(\mu_R^+) = 2E \times (0, +i \cos \theta, +1, -i \sin \theta)^\nu.$$  

(7)

Problem 1(e):
Substituting the Dirac sandwiches (5) and (7) into the pair production amplitude (1), we obtain

$$\langle \mu^-, \mu^+ | M | e_L^-, e_R^+ \rangle = \langle \mu_R^-, \mu_L^+ | M | e_R^-, e_L^+ \rangle = -e^2 \times (1 + \cos \theta),$$

$$\langle \mu_R^-, \mu_L^+ | M | e_L^-, e_R^+ \rangle = \langle \mu_L^-, \mu_R^+ | M | e_R^-, e_L^+ \rangle = -e^2 \times (1 - \cos \theta),$$

(S.12)

while all the other polarized amplitudes vanish by eqs. (3) and (4):

$$\langle \mu^-_{\text{any}}, \mu^+_{\text{any}} | M | e^-_L, e^+_R \rangle = \langle \mu^+_{\text{any}}, \mu^-_{\text{any}} | M | e^+_R, e^-_L \rangle = 0,$$

$$\langle \mu^+_L, \mu^-_L | M | e^{-}_{\text{any}}, e^+_{\text{any}} \rangle = \langle \mu^-_R, \mu^+_R | M | e^+_{\text{any}}, e^-_{\text{any}} \rangle = 0.$$  

(S.13)

The partial cross-sections (8) follow from these amplitudes according to

$$\frac{d\sigma}{d\Omega_{\text{c.m.}}} = \frac{|M|^2}{64\pi^2s} \times \left( \frac{|p'|}{|p|} = 1 \right).$$

(S.14)
Problem 1(f):
Summing the polarized cross-sections (8) over the muons’ helicities, we get
\[
\frac{d\sigma(e^{-L} + e^{+L} \rightarrow \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}} = \frac{d\sigma(e^{-R} + e^{+L} \rightarrow \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}}
= \frac{\alpha^2}{4s} \times (1 + \cos^2 \theta)^2 + \frac{\alpha^2}{4s} \times (1 - \cos^2 \theta)^2 + 0 + 0 \\
= \frac{\alpha^2}{2s} \times (1 + \cos^2 \theta)
\]
while
\[
\frac{d\sigma(e^{-L} + e^{+R} \rightarrow \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}} = \frac{d\sigma(e^{-R} + e^{+R} \rightarrow \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}}
= 0.
\]
Averaging these cross-sections over the electron’s and positron’s helicities gives
\[
\frac{d\sigma(e^{-\text{avg}} + e^{+\text{avg}} \rightarrow \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}}
= \frac{1}{4} \left( \frac{\alpha^2}{2s} \times (1 + \cos^2 \theta) + \frac{\alpha^2}{2s} \times (1 + \cos^2 \theta) + 0 + 0 \right) \\
= \frac{\alpha^2}{4s} \times (1 + \cos^2 \theta),
\]
which is exactly what we found in class for the un-polarized cross-section for \(E \gg M_\mu\).

Problem 2(a):
In the first diagram (9), the virtual photon has momentum \(q = p_1' - p_1 = p_2 - p_2'\), hence \(q^2 = t\). In the second diagram, the virtual photon’s momentum is \(\bar{q} = p_1 + p_2 = p_1' + p_2'\), hence \(\bar{q}^2 = s\). Accordingly, the two diagrams are called the \(s\)-channel diagram and the \(t\)-channel diagram.

The \(t\)-channel diagram evaluates to
\[
i\mathcal{M}_1 = -\left( \bar{v}(e^+) (ie\gamma_\mu) v(e^{+t}) \right) \times \left( \bar{u}(e^{-t}) (ie\gamma_\nu) u(e^-) \right) \times \frac{-ig^{\mu\nu}}{q^2}
= \frac{-ie^2}{t} \times \bar{v}(e^+) \gamma_\mu v(e^{+t}) \times \bar{u}(e^{-t}) \gamma^\mu u(e^-)
\]
where the overall minus sign is due to the positron-out to positron-in fermionic line. And
the $s$-channel diagram evaluates to

\[
iM_2 = +\left(\bar{v}(e^+)(ie\gamma_\mu)u(e^-)\right) \times \left(\bar{u}(e^-')(ie\gamma_\nu)v(e'^+\right) \times -\frac{ig^{\mu\nu}}{q^2} \tag{S.19}
\]

\[
= \frac{-ie^2}{s} \times \bar{v}(e^+ )\gamma_\mu u(e^-) \times \bar{u}(e^-') \gamma_\nu v(e'^+)
\]

where the overall sign is plus because both fermionic lines have an incoming or outgoing electron at one end.

**Problem 2(b):**
Summing /averaging the $|M_2|^2$ over spins works exactly as for the muon pair production discussed in class:

\[
\sum_{\text{spins}} |M_2|^2 = \left(\frac{e^2}{s}\right)^2 \sum_{\text{spins}} \left[\bar{v}(e^+)\gamma_\mu u(e^-) \times \bar{u}(e^-) \gamma_\nu v(e^+)\right] \times \left[\bar{u}(e^-') \gamma^\mu v(e'^+) \times \bar{v}(e'^+) \gamma^\nu u(e^-')\right]
\]

\[
= \left(\frac{e^2}{s}\right)^2 \text{tr}[(\not{p} - m)\gamma_\mu (\not{p} + m)\gamma_\nu] \times \text{tr}[(\not{p}' - m)\gamma^\mu (\not{p}' - m)\gamma^\nu]
\]

\[
\langle \langle \text{neglecting } m \text{ compared to the momenta} \rangle \rangle \tag{S.20}
\]

\[
\approx \left(\frac{e^2}{s}\right)^2 \text{tr}[(\not{p} - m)\gamma_\mu (\not{p} + m)\gamma_\nu] \times \text{tr}[(\not{p}' - m)\gamma^\mu (\not{p}' - m)\gamma^\nu]
\]

\[
= \left(\frac{e^2}{s}\right)^2 \times 4 \left[p_{2\mu}p_{1\nu} + p_{2\nu}p_{1\mu} - g_{\mu\nu}(p_2p_1)\right] \times 4 \left[p'_{2\mu}p'_{1\nu} + p'_{2\nu}p'_{1\mu} - g^{\mu\nu}(p'_2p'_1)\right]
\]

\[
= 16 \left(\frac{e^2}{s}\right)^2 \left[2(p'_2p_2)(p'_1p_1) + 2(p'_2p_1)(p'_1p_2) - 2(p'_2p'_1)(p_2p_1) - 2(p'_2p'_1)(p_2p_1) + 4(p'_2p'_1)(p_2p_1) \right]
\]

\[
= 32 \left(\frac{e^2}{s}\right)^2 \left[(p'_2p_2)(p'_1p_1) + (p'_2p_1)(p'_1p_2)\right]
\]

\[
= 8 \left(\frac{e^2}{s}\right)^2 [t^2 + u^2] \tag{S.20}
\]

where the last equality follows from the kinematic relations (12). Altogether,

\[
\frac{1}{4} \sum_{\text{spins}} |M_2|^2 = 2e^4 \times \frac{t^2 + u^2}{s^2}. \tag{13}
\]
Problem 2(c):
The two diagrams for Bhabha scattering are related by the crossing symmetry, so the amplitudes $M_1$ and $M_2$ are related to each other via analytic continuation of particle’s momenta. In terms of the spin-summed $|M|^2$ and Mandelstam variables,

$$\sum_{\text{spins}} |M_1(s, t, u)|^2 = \sum_{\text{spins}} |M_2(t, s, u)|^2,$$

hence eq. (13) for the second amplitude implies a similar equation for the first amplitude, but with $s$ and $t$ exchanged with each other — i.e., eq. (14).

Alternatively, we may sum the $|M_1|^2$ over all the spins in the same way as we summed the $|M_2|^2$ in part (b):

$$\sum_{\text{spins}} |M_1|^2 = \left(\frac{e^2}{t}\right)^2 \sum_{\text{spins}} \left[\bar{u}(e^-)\gamma^\mu u(e^-) \times \bar{u}(e^-)\gamma^\nu u(e^-)\right] \times \left[\bar{v}(e^+)\gamma_\mu v(e^+) \times \bar{v}(e^+)\gamma_\nu v(e^+)\right]
\approx \left(\frac{e^2}{t}\right)^2 \text{tr} \left[\not{p}_1\gamma^\mu \not{p}_2\gamma^\nu\right] \times \text{tr} \left[\not{p}_2\gamma_\mu \not{p}_1\gamma_\nu\right]
= \left(\frac{e^2}{t}\right)^2 \times 4 \left[p_1^\mu p_1^\nu + p_1^\nu p_1^\mu - g^\mu\nu(p_1^2 + m^2)\right] \times 4 \left[p_2^\mu p_2^\nu + p_2^\nu p_2^\mu - g^\mu\nu(p_2^2 + m^2)\right]
= 16 \left(\frac{e^2}{t}\right)^2 \left[2(p_1^2 p_2)(p_1 p_2) + 2(p_1^2 p_2)(p_1 p_2)\right]
= 32 \left(\frac{e^2}{t}\right)^2 \left[(p_1 p_2)(p_1 p_2) + (p_1 p_2)(p_1 p_2)\right]
= 8 \left(\frac{e^2}{t}\right)^2 \left[s^2 + u^2\right].$$

and hence

$$\frac{1}{4} \sum_{\text{spins}} |M_1|^2 = 2e^4 \times \frac{s^2 + u^2}{t^2}.$$
Problem 2(d):
The interference term between the two diagrams is more complicated:

\[ \mathcal{M}_1^* \times \mathcal{M}_2 = -\frac{e^2}{t} \left( \bar{u}(e^-)\gamma^\nu u(e^-') \times \bar{v}(e^+\gamma\mu v(e^+) \right) \times \]
\[ \frac{e^2}{s} \left( \bar{v}(e^+)\gamma\mu u(e^-) \times \bar{u}(e^-)\gamma^\nu v(e^+) \right) \]
\[ = -\frac{e^4}{st} \times \bar{u}(e^-)\gamma^\nu u(e^-') \times \bar{u}(e^-)\gamma^\mu v(e^+) \times \bar{v}(e^+)\gamma\nu v(e^+) \times \bar{v}(e^+)\gamma\mu u(e^-) \]

(S.23)

where on the second line I have re-ordered the factors so that each \( \bar{u} \) is followed by \( u \) of the same electron and each \( \bar{v} \) is followed by \( v \) for the same positron. After summing over all the spins, each \( u \times \bar{u} \) becomes \((\not{p} + m)\), each \( v \times \bar{v} \) becomes \((\not{p} - m)\), and the whole product becomes a single big trace rather than a product of two traces,

\[ \sum_{\text{spins}} \mathcal{M}_1^* \times \mathcal{M}_2 = -\frac{e^4}{st} \times \text{tr} \left[ (\not{p}_1 + m)\gamma^\nu (\not{p}_2 + m)\gamma^\mu (\not{p}_2 - m)\gamma_\nu (\not{p}_2 - m)\gamma_\mu \right] \]

(S.24)

This trace looks more complicated than it is, and we may drastically simplify it by summing over \( \nu \) and \( \mu \) before taking the trace. Back in homework set #5 we saw that

\[ \gamma^\alpha \not{p} \not{q} \gamma_\alpha = -2 \not{p} \not{q} \not{q} \quad \text{and} \quad \gamma^\alpha \not{p} \not{q} \gamma_\alpha = 4(ab). \]  

(S.25)

For the problem at hand, this gives us \( \gamma^\nu \not{p}_1^\nu \gamma^\mu \not{p}_2^\mu \gamma_\nu \not{p}_2^\nu \gamma_\mu = -2 \not{p}_2^\nu \gamma^\mu \not{p}_1^\nu \) and hence

\[ \text{tr} \left[ \not{p}_1 \gamma^\nu \not{p}_1^\nu \not{p}_2 \gamma_\nu \not{p}_2 \gamma_\mu \right] = -2 \text{tr} \left[ \not{p}_1 \not{p}_2 \gamma^\mu \not{p}_1^\nu \not{p}_2 \gamma_\mu \right] \]
\[ = -2 \text{tr} \left[ \not{p}_1 \not{p}_2 \times 4(p_1^\nu p_2^\mu) \right] \]
\[ = -8(p_1^\nu p_2^\mu) \times \text{tr} \left[ \not{p}_1 \not{p}_2 \right] \]
\[ = -8(p_1^\nu p_2^\mu) \times 4(p_1 p_2^\nu) \]
\[ = -8u^2. \]  

(S.26)

Consequently,

\[ \frac{1}{4} \sum_{\text{spins}} \mathcal{M}_1^* \times \mathcal{M}_2 = +2e^4 \times \frac{u^2}{st}. \]  

(15)
Problem 2(e):

Assembling spin sums / averages (13–15) together according to eq. (11), we get

\[
|M|^2 \overset{\text{def}}{=} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}_1 + \mathcal{M}_2|^2
\]

\[
= \frac{1}{4} \sum_{\text{spins}} \left( |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + 2 \text{Re} \mathcal{M}_1^* \mathcal{M}_2 \right)
\]

\[
= 2e^4 \times \frac{s^2 + u^2}{t^2} + 2e^4 \times \frac{t^2 + u^2}{s^2} + 4e^4 \times \frac{u^2}{st}
\]

\[
= 2e^4 \left( \frac{s^2}{t^2} + \frac{t^2}{s^2} + \frac{u^2}{s^2 t^2} \times \left( s^2 + t^2 + 2st = (s + t)^2 = u^2 \right) \right)
\]

\[
= 2e^4 \times \frac{s^4 + t^4 + u^4}{s^2 \times t^2}.
\]

Consequently, the un-polarized partial cross-section for the Bhabha scattering is

\[
\frac{d\sigma}{d\Omega_{\text{c.m.}}} = \frac{|\mathcal{M}|^2}{64\pi^2 s} = \frac{\alpha^2}{2s} \times \frac{s^4 + t^4 + u^4}{s^2 \times t^2}.
\]

To complete the problem, let’s do the kinematics. In the center of mass frame

\[
s = 4E^2 \approx 4p^2,
\]

\[
t = -(p'_1 - p_1)^2 = -2p^2(1 - \cos \theta),
\]

\[
u = -(p'_2 - p_1)^2 = -2p^2(1 + \cos \theta),
\]

hence

\[
\frac{s^4 + t^4 + u^4}{s^2 t^2} = \frac{(4p^2)^4 + (2p^2)^4 \times (1 - \cos \theta)^4 + (2p^2)^4 \times (1 + \cos \theta)^4}{(4p^2)^2 \times (2p^2)^2(1 - \cos \theta)^2}
\]

\[
= \frac{16 + (1 - \cos \theta)^4 + (1 + \cos \theta)^4}{4 \times (1 - \cos \theta)^2} = \frac{18 + 12 \cos^2 \theta + 2 \cos^4 \theta}{4 \times (1 - \cos \theta)^2}
\]

\[
= \frac{(3 + \cos^2 \theta)^2}{2(1 - \cos \theta)^2}.
\]

Plugging this formula into eq. (S.28) finally gives us

\[
\frac{d\sigma^{\text{Bhabha}}}{d\Omega_{\text{cm}}} = \frac{\alpha^2}{4s} \times \frac{(3 + \cos^2 \theta)^2}{(1 - \cos \theta)^2}.
\]

Quod erat demonstrandum.
Problem 2(a):
A point of notation: In the solutions to problem 2, the indices $\mu, e, \nu \equiv \nu_\mu$, and $\bar{\nu} \equiv \bar{\nu}_e$ denote the particles. For the Lorentz indices, I shall use $\alpha, \beta, \gamma, \delta, \kappa, \lambda, \sigma, \rho$, but never $\mu$ or $\nu$. Thus, $p_{\mu \alpha}$ denotes the $\alpha$ component of the muon’s 4–momentum, etc., etc.

Let’s start with the muon decay amplitude

$$
\mathcal{M}(\mu^- \to e^- \nu_\mu \bar{\nu}_e) = \frac{G_F}{\sqrt{2}} \left[ \bar{u}(\nu_\mu)\gamma^\alpha (1 - \gamma^5) u(\mu^-) \right] \times \left[ \bar{v}(\bar{\nu}_e)\gamma_\alpha (1 - \gamma^5) v(\bar{\nu}_e) \right].
$$

(18)

Since the Dirac conjugate of the $\gamma^\alpha (1 - \gamma^5)$ matrix is the same

$$
\gamma^\alpha (1 - \gamma^5) = (1 - \gamma^5)\gamma^\alpha = (1 + \gamma^5)\gamma^\alpha = \gamma^\alpha (1 - \gamma^5),
$$

(S.32)

The complex conjugate of the amplitude (18) is

$$
\mathcal{M}^* = \frac{G_F}{\sqrt{2}} \left[ \bar{u}(\mu^-)\gamma^\alpha (1 - \gamma^5)\gamma^\alpha u(\nu_\mu) \right] \times \left[ \bar{v}(\nu_e)\gamma_\alpha (1 - \gamma^5) u(e^-) \right],
$$

(S.33)

hence

$$
|\mathcal{M}|^2 = \frac{1}{2} G_F^2 \times \left[ \bar{u}(\nu_\mu)\gamma^\alpha (1 - \gamma^5) u(\mu^-) \right] \times \left[ \bar{v}(\bar{\nu}_e)\gamma_\alpha (1 - \gamma^5) v(\bar{\nu}_e) \right] \times \\
\times \left[ \bar{u}(\mu^-)\gamma^\beta (1 - \gamma^5) u(\nu_\mu) \right] \times \left[ \bar{v}(\nu_e)\gamma_\beta (1 - \gamma^5) u(e^-) \right] \\
= \frac{1}{2} G_F^2 \times \left[ \bar{u}(\nu_\mu)\gamma^\alpha (1 - \gamma^5) u(\mu^-) \times \bar{u}(\mu^-)\gamma^\beta (1 - \gamma^5) u(\nu_\mu) \right] \times \\
\times \left[ \bar{v}(\bar{\nu}_e)\gamma_\alpha (1 - \gamma^5) v(\bar{\nu}_e) \times \bar{v}(\nu_e)\gamma_\beta (1 - \gamma^5) u(e^-) \right].
$$

Consequently, the $|\mathcal{M}|^2$ — i.e., $|\mathcal{M}|^2$ summed over the final fermions spins and averaging over the spin of the initial muon — becomes a product of two traces,

$$
|\mathcal{M}|^2 \overset{\text{def}}{=} \frac{1}{2} \sum_{\text{all spins}} |\mathcal{M}|^2 = \frac{1}{4} G_F^2 \times \text{tr} \left( \gamma^\alpha (1 - \gamma^5)(\slashed{p}_\mu + M_\mu)\gamma^\beta (1 - \gamma^5)(\slashed{p}_e + m_e) \right) \\
\times \text{tr} \left( \gamma_\alpha (1 - \gamma^5)(\slashed{p}_\nu - m_\nu)\gamma_\beta (1 - \gamma^5)(\slashed{p}_e + m_e) \right).
$$

(S.34)

Now let’s evaluate the traces here, starting with the first:

$$
\text{tr} \left( \gamma^\alpha (1 - \gamma^5)(\slashed{p}_\mu + M_\mu)\gamma^\beta (1 - \gamma^5)(\slashed{p}_\nu + m_\nu) \right) =
$$
\[
\langle \langle \text{skipping products of odd numbers of } \gamma^\lambda \text{ matrices} \rangle \rangle \\
= \text{tr} \left( \gamma^\alpha (1 - \gamma^5) p_\mu \gamma^\beta (1 - \gamma^5) p_\nu \right) + M_\mu n_\nu \times \text{tr} \left( \gamma^\alpha (1 - \gamma^5) \gamma^\beta (1 - \gamma^5) \right)
\]

\[
\langle \langle \text{moving the } \gamma^5 \text{ through } \gamma^\lambda \text{ matrices using } \gamma^\lambda (1 + \gamma^5) = (1 + \gamma^5) \gamma^\lambda \rangle \rangle \\
= \text{tr} \left( (1 + \gamma^5) \gamma^\alpha p_\mu \gamma^\beta (1 - \gamma^5) p_\nu \right) + M_\mu m_\nu \times \text{tr} \left( (1 + \gamma^5) \gamma^\alpha \gamma^\beta (1 - \gamma^5) \right)
\]

\[
\langle \langle \text{using } (1 + \gamma^5)^2 = 2(1 + \gamma^5) \text{ while } (1 + \gamma^5)(1 - \gamma^5) = 0 \rangle \rangle \\
= 2 \text{tr} \left( (1 - \gamma^5) \gamma^\alpha p_\mu \gamma^\beta p_\nu \right) + 0
\]

\[
= 2 \text{tr} \left( \gamma^\alpha p_\mu \gamma^\beta p_\nu \right) + 2 \text{tr} \left( \gamma^5 \gamma^\alpha p_\mu \gamma^\beta p_\nu \right)
\]

\[
= 8 \left[ p_\mu^\alpha p_\nu^\beta + p_\nu^\alpha p_\mu^\beta - g^{\alpha\beta}(p_\mu \cdot p_\nu) \right] + 8 i \epsilon^{\alpha\beta\delta} p_\mu \gamma_p \gamma^\delta. \quad \text{(S.35)}
\]

Similarly, the second trace evaluates to

\[
\text{tr} \left( \gamma^\alpha (1 - \gamma^5)(p_e + m_e) \gamma^\beta (1 - \gamma^5)(p_\bar{\nu} - m_\bar{\nu}) \right) = \quad \text{(S.36)}
\]

\[
= 8 \left[ (p_e p_\bar{\nu} + p_\bar{\nu} p_e - g_{\alpha\beta}(p_e \cdot p_\bar{\nu})) \right] + 8 i \epsilon_{\alpha\beta\sigma} p_\nu^\rho p_e^\sigma.
\]

It remains to plug these two traces into eq. (S.34) and contract the Lorentz indices. Thus,

\[
|\mathcal{M}|^2 = 16 G_F^2 \times \left[ \left( p_\mu^\alpha p_\nu^\beta + p_\nu^\alpha p_\mu^\beta - g^{\alpha\beta}(p_\mu \cdot p_\nu) \right) + i \epsilon^{\alpha\gamma\beta\delta} p_\mu \gamma_p \gamma^\delta \right] \times
\]

\[
\times \left[ \left( p_{e\alpha} p_{\bar{\nu}\beta} + p_{e\beta} p_{\bar{\nu}\alpha} - g_{\alpha\beta}(p_e \cdot p_\bar{\nu}) \right) + i \epsilon_{\alpha\beta\sigma} p_\nu^\rho p_e^\sigma \right]
\]

\[
\langle \langle \text{using symmetry / antisymmetry of factors under } \alpha \leftrightarrow \beta \rangle \rangle \\
= 16 G_F^2 \left[ \left( p_\mu^\alpha p_\nu^\beta + p_\nu^\alpha p_\mu^\beta - g^{\alpha\beta}(p_\mu \cdot p_\nu) \right) \times \left( p_{e\alpha} p_{\bar{\nu}\beta} + p_{e\beta} p_{\bar{\nu}\alpha} - g_{\alpha\beta}(p_e \cdot p_\bar{\nu}) \right) \right]
\]

\[
- \epsilon^{\alpha\gamma\beta\delta} p_\mu \gamma_p \gamma^\delta \times \epsilon_{\alpha\beta\sigma} p_\nu^\rho p_e^\sigma \right]
\]

\[
\langle \langle \text{using } g^{\alpha\beta} g_{\alpha\beta} = 4 \text{ and } \epsilon^{\alpha\gamma\beta\delta} \epsilon_{\alpha\beta\sigma} = -2 \delta_\gamma^\delta \delta_\sigma^\delta + 2 \delta_\gamma^\delta \delta_\sigma^\delta \rangle \rangle \\
= 16 G_F^2 \left[ 2(p_\mu \cdot p_e)(p_\nu \cdot p_\bar{\nu}) + 2(p_\mu \cdot p_\bar{\nu})(p_\nu \cdot p_e)
\right.
\]

\[
- 2(p_\mu \cdot p_\nu)(p_e \cdot p_\bar{\nu}) - 2(p_\mu \cdot p_\bar{\nu})(p_e \cdot p_\nu)
\]

\[
+ 4(p_\mu \cdot p_\nu)(p_e \cdot p_\nu)
\]

\[
- \left[ -2(p_\mu \cdot p_\nu)(p_\nu \cdot p_e) + 2(p_\mu \cdot p_e)(p_\nu \cdot p_\nu) \right] \right)
\]
\[ Q.E.D. \]

Problem 3(b):
As explained in the Peskin & Schroeder textbook, the partial rate of a decay process (in the rest frame of the initial particle) is given by

\[ d\Gamma = \frac{1}{2M_0} \times |\mathcal{M}|^2 \times dP \]  
\[ \text{(S.37)} \]

where \( \mathcal{M} \) is the decay’s amplitude, \( |\mathcal{M}|^2 \) is \( |\mathcal{M}|^2 \) averaged over the unknown initial spins and summed over the unmeasured final spins, and \( dP \) is the infinitesimal phase space factor for the final particles. For three final particles,

\[ dP = \frac{d^3p_1}{(2\pi)^3(2E_1)} \frac{d^3p_2}{(2\pi)^3(2E_2)} \frac{d^3p_3}{(2\pi)^3(2E_3)} \times (2\pi)^3 \delta^3(p_1 + p_2 + p_3) \times (2\pi) \delta(E_1 + E_2 + E_3 - M_0) \]  
\[ \text{(S.38)} \]

where the energy-momentum conservation laws apply in the rest frame, thus \( p_1 + p_2 + p_3 = p_{\text{tot}} = 0 \) and \( E_1 + E_2 + E_3 = E_{\text{tot}} = M_0 \).

We start by using the momentum-conservation \( \delta \)–function to eliminate the \( p_3 \) as independent variable, thus

\[ dP = \frac{d^3p_1 d^3p_2}{256\pi^3} \left. \delta(E_1 + E_2 + E_3 - E_{\text{tot}}) \right|_{p_3 = -(p_1 + p_2)} \]  
\[ \text{(S.39)} \]

Next, we use spherical coordinates for the two remaining momenta,

\[ d^3p_1 = p_1^2 d\Omega_1, \quad d^3p_2 = p_2^2 d\Omega_2, \]  
\[ \text{(S.40)} \]

and then replace the \( d^2\Omega_2 \) describing the direction of the second particle’s momentum relative
to the fixed external frame with
\[
d^2 \Omega_2^{(1)} = d\theta_{12} \sin \theta_{12} d\phi_2^{(1)}
\]
describing the same direction of \(\vec{p}_2\) relative to the frame centered on \(\vec{p}_1\). Consequently,
\[
d^2 \Omega_1 \, d^2 \Omega_2 = d^2 \Omega_1 \, d^2 \Omega_2^{(1)} = \left[ d^2 \Omega_1 \, d\phi_2^{(1)} \right] \, d\theta_{12} \sin \theta_{12} \equiv d^3 \Omega \times d(\cos \theta_{12}) \quad (S.41)
\]
and hence
\[
d\mathcal{P} = \frac{d^3 \Omega}{256\pi^5} \times \frac{p_1^2 p_2^2}{E_1 E_2 E_3} \, dp_1 \, dp_2 \, d(\cos \theta_{12}) \, \delta(E_1 + E_2 + E_3 - E_{tot}) \bigg|_{\vec{p}_3 = -(\vec{p}_1 + \vec{p}_2)}. \quad (S.42)
\]
Next, we use the cosine theorem
\[
p_3^2 = (\vec{p}_1 + \vec{p}_2)^2 = p_1^2 + p_2^2 + 2p_1 p_2 \cos \theta_{12}
\]
which gives
\[
d(\cos \theta_{12}) = \frac{p_3 \, dp_3}{p_1 \, p_2}
\]
(for fixed \(p_1, p_2\)) and therefore
\[
d\mathcal{P} = \frac{d^3 \Omega}{256\pi^5} \times \frac{p_1 p_2 p_3}{E_1 E_2 E_3} \times dp_1 \, dp_2 \, dp_3 \times \delta(E_1 + E_2 + E_3 - E_{tot}). \quad (S.43)
\]
Finally, we notice that for a relativistic particle of any mass, \(p \, dp = E \, dE\) and hence
\[
d\mathcal{P} = \frac{d^3 \Omega}{256\pi^5} \times dE_1 \, dE_2 \, dE_3 \, \delta(E_1 + E_2 + E_3 - E_{tot}). \quad (S.44)
\]
Substituting this formula into eq. (S.37) gives eq. (20) for the partial decay rate.
It remains to determine the limits of kinematically allowed ways to distribute the net energy $E_{\text{tot}} = M_0$ of the process among the three final particles. Such limits follow from the triangle inequalities for the three momenta,

$$p_1 \leq p_2 + p_3, \quad p_2 \leq p_1 + p_3, \quad p_3 \leq p_1 + p_1,$$

(S.45)

which look simple in terms of momenta but produce rather complicated inequalities for the energies $E_1 = \sqrt{p_1^2 + m_1^2}$, $E_2 = \sqrt{p_2^2 + m_2^2}$, and $E_3 = \sqrt{p_3^2 + m_3^2}$. However, when all three final particles are massless, the kinematic restrictions become simply

$$E_1 \leq E_2 + E_3 = M - E_1,$$

$$E_2 \leq E_1 + E_3 = M - E_2,$$

$$E_3 \leq E_1 + E_2 = M - E_3,$$

(S.46)

where the second expression on each right hand side follows from the net energy conservation $E_1 + E_2 + E_3 = M$. In other words, the kinematically allowed energies of the three final particles’ range over

$$0 \leq E_1, E_2, E_3 \leq \frac{1}{2}M_0, \quad \text{while} \quad E_1 + E_2 + E_3 = M_0.$$  

(21)

The picture below shows this range in the $(E_1, E_2, E_3)$ space:
**Problem 3(c):**

In the muon’s rest frame

\[
(p_\mu \cdot p_\nu) = M_\mu E_{\bar{\nu}}
\]

while

\[
(p_e \cdot p_e) = E_e E_{\nu} - p_e p_{\nu} \cos \theta_{e\nu}
\]

\[
= E_e E_{\nu} + \frac{1}{2} p_e^2 + \frac{1}{2} p_{\nu}^2 - \frac{1}{2} p_{\bar{\nu}}^2
\]

\[
\approx E_e E_{\nu} + \frac{1}{2} E_e^2 + \frac{1}{2} E_{\nu}^2 - \frac{1}{2} E_{\bar{\nu}}^2
\]

\[
= \frac{1}{2} (E_e + E_{\nu})^2 - \frac{1}{2} E_{\bar{\nu}}^2
\]

\[
= \frac{1}{2} (M_\mu - E_{\bar{\nu}})^2 - \frac{1}{4} E_{\bar{\nu}}^2
\]

\[
= \frac{1}{2} M_\mu (M_\mu - 2E_{\bar{\nu}}),
\]

so the spin-averages muon decay amplitude \(^2\) (19) becomes

\[
|\mathcal{M}|^2 = 32 G_F^2 M_\mu^2 E_{\bar{\nu}} (M_\mu - 2E_{\bar{\nu}}).
\]

Consequently, eq. (20) gives us the partial decay of the muon at rest as

\[
d\Gamma(\mu^- \to e^- \nu_\mu \bar{\nu}_e) = \frac{G_F^2}{16 \pi^3} M_\mu E_{\bar{\nu}} (M_\mu - 2E_{\bar{\nu}}) \times dE_e dE_{\nu} dE_{\bar{\nu}} d^3 \Omega (E_e + E_{\nu} + E_{\bar{\nu}} - M_\mu).
\]

(S.51)

At this point we are ready to integrate over the final-state variables. In light of \( \int d^3 \Omega = 8 \pi^2 \) and the kinematic limits (21), we immediately obtain

\[
\Gamma = \frac{G_F^2 M_\mu}{2 \pi^3} \int_0^{\frac{1}{2} M_\mu} dE_e \int_0^{\frac{1}{2} M_\mu} dE_{\bar{\nu}} \int_0^{\frac{1}{2} M_\mu} dE_{\nu} E_{\bar{\nu}} (M_\mu - 2E_{\bar{\nu}}) \delta(E_e + E_{\nu} + E_{\bar{\nu}} - M_\mu)
\]

\[
= \frac{G_F^2 M_\mu}{2 \pi^3} \int_0^{\frac{1}{2} M_\mu} dE_e \int_0^{\frac{1}{2} M_\mu} dE_{\bar{\nu}} \int_0^{\frac{1}{2} M_\mu} dE_{\nu} (M_\mu - 2E_{\bar{\nu}}) \times \text{restrict to}(E_{\nu} = M - E_e - E_{\bar{\nu}} \leq \frac{1}{2} M)
\]

\[
= \frac{G_F^2 M_\mu}{2 \pi^3} \int_0^{\frac{1}{2} M_\mu} dE_e \int_0^{\frac{1}{2} M_\mu} dE_{\bar{\nu}} E_{\bar{\nu}} (M_\mu - 2E_{\bar{\nu}})
\]

\[
= \frac{G_F^2 M_\mu}{2 \pi^3} \int_0^{\frac{1}{2} M_\mu} dE_e E_e^2 (\frac{1}{2} M_\mu - \frac{2}{3} E_e).
\]

(S.52)
In other words, the partial muon decay rate with respect to the final electron’s energy is given by

$$\frac{d\Gamma}{dE_e} = \frac{G_F^2 M_\mu}{12\pi^3} \times E_e^2 (3M_\mu - 4E_e)$$

(S.53)

or rather

$$\frac{d\Gamma}{dE_e} \approx \begin{cases} \frac{G_F^2}{12\pi^3} M_\mu E_e^2 (3M_\mu - 4E_e) & \text{for } E_e < \frac{1}{2} M_\mu, \\ 0 & \text{for } E_e > \frac{1}{2} M_\mu. \end{cases}$$

(S.54)

Graphically,

Note how this curve smoothly reaches its maximum at $E_e = \frac{1}{2} M_\mu$ and then abruptly falls down to zero.

It remains to calculate the total decay rate of the muon by integrating the partial rate (S.54) over the electron’s energy. The result is

$$\Gamma_{tot}(\mu \rightarrow e\nu\bar{\nu}) = \frac{G_F^2 M_\mu}{12\pi^3} \times \int_0^{\frac{1}{2} M_\mu} dE_e E_e^2 (3M_\mu - 4E_e) = \frac{G_F^2 M_\mu^5}{192\pi^3}.$$  

(S.55)