Dirac Trace Techniques

Consider a QED amplitude involving one incoming electron with momentum $p$ and spin $s$, one outgoing electron with momentum $p'$ and spin $s'$, and some photons. There may be several Feynman diagrams contributing to this amplitude, but they all have the same external legs and the corresponding factors $u(p, s)$ and $\bar{u}(p', s')$. Consequently, we may write the amplitude as

$$\langle e^{-}, \ldots | M | e^{-}, \ldots \rangle = \bar{u}(p', s')\Gamma u(p, s)$$ \hspace{1cm} (1)

where $\Gamma$ is comprises all the other factors of the QED Feynman rules; for the moment, we don’t want to be specific, so $\Gamma$ is just some kind of a $4 \times 4$ matrix.

In many experiments, the initial electrons come in un-polarized beam, 50% having spin $s = +\frac{1}{2}$ and 50% having $s = -\frac{1}{2}$. At the same time, the detector of the final electrons measures their momenta $p'$ but is blind to their spins $s'$. The cross-section $\sigma$ measured by such an experiment would be the average of the polarized cross-sections $\sigma(s, s')$ with respect to initial spins $s$ and the sum over the final spins $s'$, thus

$$\sigma = \frac{1}{2} \sum_s \sum_{s'} \sigma(s, s').$$ \hspace{1cm} (2)

Similar averaging / summing rules apply to the un-polarized partial cross-sections,

$$\overline{\frac{d\sigma}{d\Omega}} = \frac{1}{2} \sum_s \sum_{s'} \frac{d\sigma(s, s')}{d\Omega},$$ \hspace{1cm} (3)

etc., etc. Since all total or partial cross-sections are proportional to mod-squares $|M|^2$ of amplitudes $M$, we need to know how to calculate

$$|M|^2 \overset{\text{def}}{=} \frac{1}{2} \sum_s \sum_{s'} |M(s, s')|^2$$ \hspace{1cm} (4)

for amplitudes such as (1).
To do such a calculation efficiently, we need to recall two things about Dirac spinors. First,

\[ M = \bar{u}(p', s') \Gamma u(p, s) \quad \text{then} \quad M^* = \bar{u}(p, s) \Gamma u(p', s') \]  

(5)

where \( \Gamma = \gamma^0 \Gamma^\dagger \gamma^0 \) is the Dirac conjugate of the matrix \( \Gamma \); for a product \( \gamma^\lambda \cdots \gamma^\nu \) of Dirac matrices, \( \bar{\gamma} \cdots \gamma^\nu = \gamma^\nu \cdots \gamma^\lambda \). Second,

\[ \sum_{s} u_\alpha(p, s) \times \bar{u}_\beta(p, s) = (p + m)_{\alpha \beta} \]  

(6)

and likewise

\[ \sum_{s'} u_\gamma(p', s') \times \bar{u}_\delta(p', s') = (p' + m)_{\gamma \delta} \]  

(7)

Combining these two facts, we obtain

\[ \sum_{s, s'} |M = \bar{u}(p', s') \Gamma u(p, s)|^2 = \sum_{s, s'} \bar{u}(p', s') \Gamma u(p, s) \times \bar{u}(p, s) \Gamma u(p', s') \]

\[ = \sum_{s, s'} \sum_{\delta, \alpha} \bar{u}_\delta(p', s') \Gamma_{\delta \alpha} u_\alpha(p, s) \times \sum_{\beta, \gamma} \bar{u}_\beta(p, s) \Gamma_{\beta \gamma} u_\gamma(p', s') \]

\[ = \sum_{\alpha, \beta, \gamma, \delta} \Gamma_{\delta \alpha} \Gamma_{\beta \gamma} \times \left( \sum_{s} u_\alpha(p, s) \bar{u}_\beta(p, s) \right) \times \left( \sum_{s'} u_\gamma(p', s') \bar{u}_\delta(p', s') \right) \]

\[ = \sum_{\alpha, \beta, \gamma, \delta} \Gamma_{\delta \alpha} \Gamma_{\beta \gamma} \times (p + m)_{\alpha \beta} \times (p' + m)_{\gamma \delta} \]

\[ = \sum_{\gamma} \left( \text{matrix product} (p' + m) \Gamma (p + m) \Gamma \right)_{\gamma \gamma} \]

\[ = \text{tr} \left( (p' + m) \Gamma (p + m) \Gamma \right) \]  

(8)

and hence

\[ \langle e^-, \ldots | M | e^-, \ldots \rangle = \bar{u}(p', s') \Gamma u(p, s) \quad \Rightarrow \quad \frac{1}{2} \sum_{s, s'} |M|^2 = \frac{1}{2} \text{tr} \left( (p' + m) \Gamma (p + m) \Gamma \right). \]  

(9)

A similar trace formula exists for un-polarized scattering of positrons. In this case, the
amplitude is
\[
\langle e^+, \ldots | \mathcal{M} | e^+, \ldots \rangle = \bar{v}(p, s) \Gamma v(p', s')
\] (10)

(note that \(v(p', s')\) belongs to the outgoing positron while \(\bar{v}(p, s)\) belongs to the incoming \(s^+\)), and we need to average \(|\mathcal{M}|^2\) over \(s\) and sum over \(s'\). Using
\[
\sum_s v_\alpha(p, s) \times \bar{v}_\beta(p, s) = (\not{p} - m)_{\alpha\beta}
\] (11)

and working through algebra similar to eq. (8), we arrive at
\[
\langle e^+, \ldots | \mathcal{M} | e^+, \ldots \rangle = \bar{v}(p, s) \Gamma v(p', s') \implies \frac{1}{2} \sum_{s, s'} |\mathcal{M}|^2 = \frac{1}{2} \operatorname{tr} \left( (\not{p} - m) \Gamma (\not{p}' - m) \Gamma \right).
\] (12)

Now suppose an electron with momentum \(p_1\) and spin \(s_1\) and a positron with momentum \(p_2\) and spin \(s_2\) come in and annihilate each other. In this case, the amplitude has form
\[
\langle \ldots | \mathcal{M} | e^-_1, e^+_2, \ldots \rangle = \bar{v}(p_2, s_2) \Gamma u(p_1, s_1)
\] (13)

for some \(\Gamma\), and if both electron and positron beams are un-polarized, we need to average the \(|\mathcal{M}|^2\) over both spins \(s_1\) and \(s_2\). Again, there is a trace formula for such averaging, namely
\[
\langle \ldots | \mathcal{M} | e^-_1, e^+_2, \ldots \rangle = \bar{v}(p_2, s_2) \Gamma u(p_1, s_1) \implies \frac{1}{4} \sum_{s_1, s_2} |\mathcal{M}|^2 = \frac{1}{4} \operatorname{tr} \left( (\not{p}_2 - m) \Gamma (\not{p}_1 + m) \Gamma \right).
\] (14)

Finally, for a process in which an electron-positron pair is created, the amplitude has form
\[
\langle e^-_1, e^+_2, \ldots | \mathcal{M} | \ldots \rangle = \bar{u}(p'_1, s'_1) \Gamma v(p'_2, s'_2),
\] (15)

and if we do not detect the spins of the outgoing electron and positron but only their momenta, then we should sum the \(|\mathcal{M}|^2\) over both spins \(s_1\) and \(s_2\). Again, there is a trace
formula for this sum, namely

\[ \langle e_1', e_2', \ldots, | M | \ldots \rangle = \bar{u}(p_1', s_1') \Gamma_v(p_2', s_2') \implies \sum_{s_1, s_2} |M|^2 = \text{tr} \left( (p_1' + m) \Gamma (p_2' - m) \Gamma \right) \]

(16)

Processes involving 4 or more un-polarized fermions also have trace formulae for the spin sums. For example, consider an \( e^- + e^+ \) collision in which a muon pair \( \mu^- + \mu^+ \) is created. There is one tree diagram for this process,

\[ i \langle \mu^-, \mu^+ | M | e^-, e^+ \rangle = \frac{-ig^{\lambda \nu}}{q^2} \times \bar{u}(\mu^-)(i e^\gamma_{\lambda}) v(\mu^+) \times \bar{v}(e^+)(i e^\gamma_{\nu}) u(e^-) \]

\[ = \frac{ie^2}{s} \times \bar{u}(\mu^-) \gamma^\nu v(\mu^+) \times \bar{v}(e^+) \gamma^\nu u(e^-) \]

(18)

where

\[ s = q^2 = (p_1 + p_2)^2 = (p_1' + p_2')^2 = E_{c.m.}^2 \]

(19)

is the square of the total energy in the center-of-mass frame. When the initial electron and positron are both un-polarized, and we do not measure the spins of the muons but only their moments, we need to average the \(|M|^2\) over both initial spins \( s_1, s_2 \) and some over both final spins \( s_1', s_2' \), thus

\[ \left( \frac{d\sigma}{d\Omega} \right)_{c.m.} = \frac{|M|^2}{64\pi^2 s} \quad \text{where} \quad |M|^2 \equiv \frac{1}{4} \sum_{s_1, s_2, s_1', s_2'} |M|^2. \]

(20)
For the amplitude (18) at hand,

\[
\mathcal{M} \times \mathcal{M}^* = \frac{e^4}{s^2} \times \left( \bar{u}(\mu^-)\gamma^\nu v(\mu^+) \times \bar{v}(e^+)\gamma_\nu u(e^-) \right) \times \left( \bar{v}(\mu^+)\gamma^\lambda u(\mu^-) \times \bar{u}(e^-)\gamma_\lambda v(e^+) \right)
\]

— note sums over two separate Lorentz indices \( \lambda \) and \( \nu \) —

\[
= \frac{e^4}{s^2} \times \left( \bar{u}(\mu^-)\gamma^\nu v(\mu^+) \times \bar{v}(\mu^+)\gamma^\lambda u(\mu^-) \right) \times \left( \bar{v}(e^+)\gamma_\nu u(e^-) \times \bar{u}(e^-)\gamma_\lambda v(e^+) \right)
\]

(21)

and consequently

\[
|\mathcal{M}|^2 = \frac{e^4}{4s^2} \times \left( \sum_{s_1, s_2} \bar{u}(\mu^-)\gamma^\nu v(\mu^+) \times \bar{v}(\mu^+)\gamma^\lambda u(\mu^-) \right) \times \left( \sum_{s_1, s_2} \bar{v}(e^+)\gamma_\nu u(e^-) \times \bar{u}(e^-)\gamma_\lambda v(e^+) \right)
\]

(22)

Calculating Dirac Traces

Thus far, we have learned how to express un-polarized cross-sections in terms of Dirac traces (\( i.e. \), traces of products of the Dirac \( \gamma^\lambda \) matrices). In this section, we shall learn how to calculate such traces.

Dirac traces do not depend on the specific form of the \( \gamma^0, \gamma^1, \gamma^2, \gamma^4 \) matrices but are completely determined by the Clifford algebra

\[
\{ \gamma^\mu, \gamma^\nu \} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu \nu}.
\]

(23)

To see how this works, please recall the key property of the trace of any matrix product:

\* \( \text{tr}(AB) = \text{tr}(BA) \) for any two matrices \( A \) and \( B \). Proof:

\[
\text{tr}(AB) = \sum_\alpha (AB)_{\alpha \alpha} = \sum_\alpha \sum_\beta A_{\alpha \beta} B_{\beta \alpha} = \sum_\beta (BA)_{\beta \beta} = \text{tr}(BA).
\]

(24)

This symmetry has two important corollaries:
• All commutators have zero traces, $\text{tr}([A, B]) = 0$ for any $A$ and $B$.

• Traces of products of several matrices have cyclic symmetry

$$\text{tr}(ABC \cdots YZ) = \text{tr}(BC \cdots YZA) = \text{tr}(C \cdots YZAB) = \cdots = \text{tr}(ZABC \cdots Y).$$

(25)

Using these properties it is easy to show that

$$\text{tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu} \implies \text{tr}(a b) = 4(ab) \equiv 4a_\mu b^\mu.$$  

(26)

Indeed,

$$\text{tr}(\gamma^\mu \gamma^\nu) = \text{tr}(\gamma^\nu \gamma^\mu) = \text{tr}(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}) = \text{tr}(g^{\mu\nu}) = g^{\mu\nu} \times \text{tr}(1) = g^{\mu\nu} \times 4,$$

(27)

where the last equality follows from Dirac matrices being $4 \times 4$ and hence

$$\text{tr}(1) = 4.$$  

(28)

Next, all products of any odd numbers of the $\gamma^\mu$ matrices have zero traces,

$$\text{tr}(\gamma^\mu) = 0, \quad \text{tr}(\gamma^\lambda \gamma^\mu \gamma^\nu) = 0, \quad \text{tr}(\gamma^\lambda \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 0, \quad \text{etc.},$$

(29)

and hence

$$\text{tr}(a) = 0, \quad \text{tr}(a b c) = 0, \quad \text{tr}(a b c d e) = 0, \quad \text{etc.}$$

(30)

To see how this works, we can use the $\gamma^5$ matrix which anticommutes with all the $\gamma^\mu$ and hence with any product $\Gamma$ of an odd number of the Dirac $\gamma$'s, $\gamma^5 \Gamma = -\Gamma \gamma^5$. Consequently,

$$\Gamma = \gamma^5 \gamma^5 \Gamma = -\gamma^5 \Gamma \gamma^5 = -\frac{1}{2}[\gamma^5 \Gamma, \gamma^5]$$

(31)

and hence

$$\text{tr}(\Gamma) = -\frac{1}{2} \text{tr}([\gamma^5 \Gamma, \gamma^5]) = 0.$$  

(32)

Products of even numbers $n = 2m$ of $\gamma$ matrices have non-trivial traces, and we may calculate them recursively in $n$. We already know the traces for $n = 0$ and $n = 2$, so consider
a product $\gamma^k\gamma^\lambda\gamma^\mu\gamma^\nu$ of $n = 4$ matrices. Thanks to the cyclic symmetry of the trace, 

$$\text{tr}(\gamma^k\gamma^\lambda\gamma^\mu\gamma^\nu) = \text{tr}(\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^k) = \text{tr}\left(\frac{1}{2}\{\gamma^k, \gamma^\lambda\gamma^\mu\gamma^\nu\}\right)$$

(33)

where the anticommutator follows from the Clifford algebra (23):

$$\frac{1}{2}\{\gamma^k, \gamma^\lambda\gamma^\mu\gamma^\nu\} = g^{k\lambda} \times \gamma^\mu\gamma^\nu - g^{k\mu} \times \gamma^\lambda\gamma^\nu + g^{k\nu} \times \gamma^\lambda\gamma^\mu,$$

(34)

cf. homework set #5. Therefore,

$$\text{tr}(\gamma^k\gamma^\lambda\gamma^\mu\gamma^\nu) = g^{k\lambda} \times \text{tr}(\gamma^\mu\gamma^\nu) - g^{k\mu} \times \text{tr}(\gamma^\lambda\gamma^\nu) + g^{k\nu} \times \text{tr}(\gamma^\lambda\gamma^\mu)$$

(35)

and hence

$$\text{tr}(\not\!a\not\!b\not\!c\not\!d) = 4(ab)(cd) - 4(ac)(bd) + 4(ad)(bc).$$

(36)

Note that in eq. (35) we have expressed the trace of a 4–$\gamma$ product to traces of 2–$\gamma$ products. Similar recursive formulae exist for all even numbers of $\gamma$ matrices,

$$\text{tr}(\gamma^{\nu_1}\gamma^{\nu_2}\cdots\gamma^{\nu_n}) = \text{tr}\left(\frac{1}{2}\{\gamma^{\nu_1}, \gamma^{\nu_2}\cdots\gamma^{\nu_n}\}\right)$$

for even $n$

$$= \sum_{k=2}^{n} (-1)^k g^{\nu_1\nu_k} \times \text{tr}\left(\gamma^{\nu_2}\cdots\not\!\times^{k}\cdots\gamma^{\nu_n}\right).$$

(37)

For example, for $n = 6$

$$\text{tr}(\gamma^k\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = g^{k\lambda} \times \text{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) - g^{k\mu} \times \text{tr}(\gamma^\lambda\gamma^\nu\gamma^\rho\gamma^\sigma) + g^{k\nu} \times \text{tr}(\gamma^\lambda\gamma^\mu\gamma^\rho\gamma^\sigma)$$

$$= 4g^{k\lambda} \times \left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)$$

$$- 4g^{k\mu} \times \left(g^{\lambda\nu}g^{\rho\sigma} - g^{\lambda\rho}g^{\nu\sigma} + g^{\lambda\sigma}g^{\nu\rho}\right)$$

$$+ 4g^{k\nu} \times \left(g^{\lambda\mu}g^{\rho\sigma} - g^{\lambda\rho}g^{\mu\sigma} + g^{\lambda\sigma}g^{\mu\rho}\right)$$

$$- 4g^{k\rho} \times \left(g^{\lambda\mu}g^{\nu\sigma} - g^{\lambda\nu}g^{\mu\sigma} + g^{\lambda\sigma}g^{\mu\nu}\right)$$

$$+ 4g^{k\sigma} \times \left(g^{\lambda\mu}g^{\nu\rho} - g^{\lambda\nu}g^{\mu\rho} + g^{\lambda\rho}g^{\mu\nu}\right).$$

(38)

For products of more $\gamma$ matrices, the recursive formulae (37) for traces produce even more terms (105 terms for $n = 8$, 945 terms for $n = 10$, etc., etc.), so it helps to reduce
Whenever possible. For example, if the matrix product inside the trace contains two $\varphi$ matrices (for the same 4-vector $a^{\mu}$) next to each other, you can simplify the product using $\varphi \varphi = a^{2}$, thus
\[
\text{tr}(\varphi \varphi \cdot \cdot \cdot \varphi) = a^{2} \times \text{tr}(\cdot \cdot \cdot \varphi) .
\]
(39)

Also, when a product contains $\gamma^{\alpha}$ and $\gamma_{\alpha}$ with the same Lorentz index $\alpha$ which should be summed over, we may simplify the trace using
\[
\gamma^{\alpha} \gamma_{\alpha} = 4, \quad \gamma^{\alpha} \varphi \gamma_{\alpha} = -2 \varphi, \quad \gamma^{\alpha} \varphi \gamma_{\alpha} = +4(ab), \quad \gamma^{\alpha} \varphi \varphi \gamma_{\alpha} = -2 \varphi \varphi,
\]
(40)
e tc., cf. homework set #5.

In the electroweak theory, one often needs to calculate traces of products containing the $\gamma^{5}$ matrix. If the $\gamma^{5}$ appears more than once, we may simplify the product using $\gamma^{5} \gamma^{5} = 1$ and $\gamma^{5} \gamma^{\nu} = -\gamma^{\nu} \gamma^{5}$. For example,
\[
\text{tr}\left(\gamma^{\mu}(1 - \gamma^{5}) \slashed{\nu} \gamma^{\nu}(1 - \gamma^{5}) \varphi\right) = \text{tr}\left(\gamma^{\mu}(1 - \gamma^{5}) \slashed{\nu} (1 + \gamma^{5}) \gamma^{\nu} \varphi\right) = \text{tr}\left(\gamma^{\mu}(1 - \gamma^{5})(1 - \gamma^{5}) \slashed{\nu} \gamma^{\nu} \varphi\right) = 2 \text{tr}\left(\gamma^{\mu}(1 - \gamma^{5}) \slashed{\nu} \gamma^{\nu} \varphi\right)
\]
\[
= 2 \text{tr}\left((1 + \gamma^{5}) \gamma^{\mu} \slashed{\nu} \gamma^{\nu} \varphi\right) = 2 \text{tr}\left(\gamma^{\mu} \slashed{\nu} \gamma^{\nu} \varphi\right) + 2 \text{tr}\left(\gamma^{5} \gamma^{\mu} \slashed{\nu} \gamma^{\nu} \varphi\right).
\]
(41)

When the $\gamma^{5}$ appears just one time, we may use $\gamma^{5} = i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}$ to show that
\[
\text{tr}(\gamma^{5}) = 0, \quad \text{tr}(\gamma^{5} \gamma^{\nu}) = 0, \quad \text{tr}(\gamma^{5} \gamma^{\mu} \gamma^{\nu}) = 0, \quad \text{tr}(\gamma^{5} \gamma^{\lambda} \gamma^{\mu} \gamma^{\nu}) = 0,
\]
(42)

while
\[
\text{tr}(\gamma^{5} \gamma^{\kappa} \gamma^{\lambda} \gamma^{\mu} \gamma^{\nu}) = -4i\varepsilon^{\kappa\lambda\mu\nu}.
\]
(43)

For more $\gamma^{\nu}$ matrices accompanying the $\gamma^{5}$ we have
\[
\text{tr}(\gamma^{5} \gamma^{\nu_{1}} \cdot \cdot \cdot \gamma^{\nu_{n}}) = 0 \quad \forall \text{ odd } n,
\]
(44)
while for even \( n = 6, 8, \ldots \) there are recursive formulae based on the identity

\[
\gamma^5 \gamma^\lambda \gamma^\mu \gamma^\nu = g^\lambda \mu \times \gamma^5 \gamma^\nu - g^\lambda \nu \times \gamma^5 \gamma^\mu + g^\mu \nu \times \gamma^5 \gamma^\lambda - i \epsilon^{\lambda \mu \nu \rho} \times \gamma_\rho. \tag{45}
\]

**Muon Pair Production**

As an example of trace technology, let us calculate the traces (22) for the muon pair production. Let’s start with the trace due to summing over muons’ spins,

\[
\text{tr} \left( (p'_1 + M_\mu) \gamma^\lambda (p'_2 - M_\mu) \gamma^\nu \right) = \text{tr}(p'_1 \gamma^\lambda p'_2 \gamma^\nu) + M_\mu \times \text{tr}(\gamma^\lambda p'_2 \gamma^\nu) - M_\mu \times \text{tr}(p'_1 \gamma^\lambda \gamma^\nu) - M^2_\mu \times \text{tr}(\gamma^\lambda \gamma^\nu). \tag{46}
\]

In the second line here, we have three \( \gamma \) matrices inside each trace, so those traces vanish. In the third line, \( \text{tr}(\gamma^\lambda \gamma^\nu) = 4g^{\lambda \nu} \). Finally, the trace in the first line follows from eq. (35),

\[
\text{tr}(p'_1 \gamma^\lambda p'_2 \gamma^\nu) = p'_{\alpha \alpha} p'_{\beta \beta} \times \text{tr}(\gamma^\alpha \gamma^\lambda \gamma^\beta \gamma^\nu) = p'_{\alpha \alpha} p'_{\beta \beta} \times 4 \left( g^{\alpha \lambda} \times g^{\beta \nu} - g^{\alpha \beta} \times g^{\lambda \nu} + g^{\alpha \nu} \times g^{\lambda \beta} \right) \tag{47}
\]

\[
= 4p'^{\lambda \nu} - 4(p'_1 p'_2) \times g^{\lambda \nu} + 4p'^{\nu \lambda} \times p'_2.
\]

Altogether,

\[
\text{tr} \left( (p'_1 + M_\mu) \gamma^\lambda (p'_2 - M_\mu) \gamma^\nu \right) = 4p'^{\lambda \nu} - 4p'^{\nu \lambda} + 4p'_{\lambda \mu} + 4(p'_1 p'_2) \times g^{\lambda \nu} = 4p'^{\lambda \nu} + 4p'^{\nu \lambda} - 2s \times g^{\lambda \nu} \tag{48}
\]

where on the last line I have used

\[
s \equiv (p'_1 + p'_2)^2 = p'^2_1 + p'^2_2 + 2(p'_1 p'_2) = 2M^2_\mu + 2(p'_1 p'_2). \tag{49}
\]
Similarly, for the second trace (22) due to averaging over electron’s and positron’s spins, we have
\[
\text{tr}
\left( (p_2' - m_e) \gamma_\nu (p_1 + m_e) \gamma_\lambda \right) = \text{tr}(p_2' \gamma_\nu \gamma_\lambda) + m_e \times \text{tr}(p_2' \gamma_\nu \gamma_\lambda) - m_e \times \text{tr}(\gamma_\nu p_1 \gamma_\lambda) - m_e^2 \times \text{tr}(\gamma_\nu \gamma_\lambda) 
\]
\[
= 4p_2' \times p_1 \lambda - 4p_2p_1 \times g_{\nu \lambda} + 4p_2 \lambda \times p_1 \nu + m_e \times 0 - m_e \times 0 - m_e^2 \times 4g_{\nu \lambda} 
\]
\[
= 4p_2'p_1 \lambda + 4p_2 \lambda p_1 \nu - 4((p_2p_1) + m_e^2) \times g_{\lambda \nu} 
\]
\[
= 4p_2'p_1 \lambda + 4p_2 \lambda p_1 \nu - 2s \times g_{\lambda \nu} .
\]

where on the last line I have used
\[
s \equiv (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2(p_2p_1) = 2m_e^2 + 2(p_2p_1). \quad (51)
\]

It remains to multiply the two traces and sum over the Lorentz indices \( \lambda \) and \( \nu \):
\[
\text{tr}
\left( (p_1' + M_\mu) \gamma_\nu (p_2' - M_\mu) \gamma_\lambda \right) \times \text{tr}
\left( (p_2' - m_e) \gamma_\nu (p_1 + m_e) \gamma_\lambda \right) = 
\]
\[
= \left( 4p_1' \lambda p_2' \mu \lambda + 4p_1' \lambda p_2' \mu \nu - 2s \times g_{\nu \lambda} \right) \times 
\left( 4p_2p_1 \nu + 4p_2 \lambda p_1 \nu - 2s \times g_{\nu \lambda} \right) 
\]
\[
= 16(p_1' \lambda p_2' \mu \lambda + p_2' \lambda p_1 \nu) \times 
\left( p_2p_1 \lambda + p_2 \lambda p_1 \nu \right) - 8s g_{\nu \lambda} \times (p_2p_1 \nu + p_2 \lambda p_1 \nu) 
\]
\[
- 8s g_{\nu \lambda} \times (p_1' \lambda p_2' \nu + p_1' \nu p_2' \lambda) + 4s^2 g_{\lambda \nu} 
\]
\[
= 16 \times 2 \times \left( (p_1'p_1)(p_2'p_2) + (p_2'p_1)(p_1'p_2) \right) 
\]
\[
- 8s \times 2(p_1p_2) - 8s \times 2(p_1'p_2') + 4s^2 \times 4 
\]
\[
= 32(p_1'p_1)(p_2'p_2) + 32(p_2'p_1)(p_1'p_2) + 16s \times (M_\mu^2 + m_e^2) \quad (52)
\]

where the last line follows from eqs. (49) and (51). Hence, in eq. (22) we have
\[
|\mathcal{M}|^2 = \frac{1}{4} \sum_{\text{all spins}} |\mathcal{M}|^2 = \frac{4e^4}{s^2} \times \left( 2(p_1'p_1)(p_2'p_2) + 2(p_2'p_1)(p_1'p_2) + s(M_\mu^2 + m_e^2) \right). \quad (53)
\]
Finally, let’s work out the kinematics of pair production. In the center-of-mass frame, 
\( p_{1,2}' = (E, \pm \mathbf{p}) \) and \( p_{1,2}'' = (E, \mathbf{p'}) \), same \( E = \frac{1}{2}E_{\text{cm}} \) but \( \mathbf{p'} \neq \mathbf{p} \). Therefore,

\[
(p'_1 p_1) = (p'_2 p_2) = E^2 - \mathbf{p'} \cdot \mathbf{p},
(p'_2 p_1) = (p'_1 p_2) = E^2 + \mathbf{p'} \cdot \mathbf{p},
\]

\[
s = 4E^2,
2(p'_1 p_1)(p'_2 p_2) + 2(p'_2 p_1)(p'_1 p_2) = 2(E^2 - \mathbf{p'} \cdot \mathbf{p})^2 + 2(E^2 + \mathbf{p'} \cdot \mathbf{p})^2
= 4E^4 + 4(\mathbf{p'} \cdot \mathbf{p})^2
= 4E^4 + 4\mathbf{p'}^2 \mathbf{p}^2 \times \cos^2 \theta,
\]

and consequently

\[
|\mathcal{M}|^2 = e^4 \left( 1 + \frac{\mathbf{p'}^2 \mathbf{p}^2}{E^4} \times \cos^2 \theta + \frac{M^2_{\mu} + m^2_e}{E^2} \right),
\]

where \( \mathbf{p'}^2 = E^2 - M^2_{\mu} \) and \( \mathbf{p}^2 = E^2 - m^2_e \).

We may simplify this expression a bit using the experimental fact that the muon is much heavier than the electron, \( M_{\mu} \approx 207m_e \), so we need ultra-relativistic \( e^\mp \) to produce \( \mu^\mp \), \( E > M_{\mu} \gg m_e \). This allows us to neglect the \( m^2_e \) term in eq. (55) and let \( \mathbf{p}^2 = E^2 \), thus

\[
|\mathcal{M}|^2 = e^4 \left( 1 + \frac{M^2_{\mu}}{E^2} \right) \left( 1 + \frac{M^2_{\mu}}{E^2} \right) \times \cos^2 \theta \times \sqrt{1 - \frac{M^2_{\mu}}{E^2}},
\]

and consequently the partial cross-section is

\[
\left( \frac{d\sigma}{d\Omega} \right)_{\text{c.m.}} = \frac{\alpha^2}{4s} \times \left( 1 + \frac{M^2_{\mu}}{E^2} \right) \left( 1 + \frac{M^2_{\mu}}{E^2} \right) \times \cos^2 \theta \times \sqrt{1 - \frac{M^2_{\mu}}{E^2}}
\]

where the root comes from the phase-space factor \( |\mathbf{p'}|/|\mathbf{p}| \) for inelastic processes.

Looking at the angular dependence of this partial cross-section, we see that just above the energy threshold, for \( E = M_{\mu} + \text{small} \), the muons are produced isotropically in all directions.
On the other hand, for very high energies $E \gg M_\mu$ when all 4 particles are ultra-relativistic,

\[
\left( \frac{d\sigma}{d\Omega} \right)_{\text{c.m.}} \propto 1 + \cos^2 \theta. \tag{58}
\]

In homework set #11 you will see that the polarized cross sections depend on the angle as $(1 \pm \cos \theta)^2$ where the sign $\pm$ depends on the helicities of initial and final particles; for the un-polarized particles, we average / sum over helicities, and that produces the averaged angular distribution $1 + \cos^2$.

Finally, the total cross-section of muon pair production follows from eq. (57) using

\[
\int d^2\Omega = 4\pi, \quad \int d^2\Omega \cos^2 \theta = \frac{4\pi}{3}, \tag{59}
\]

hence

\[
\sigma_{\text{tot}}(e^-e^+ \rightarrow \mu^-\mu^+) = \frac{4\pi}{3} \frac{\alpha^2}{s} \times \left( 1 + \frac{M_\mu^2}{2E^2} \right) \sqrt{1 - \frac{M_\mu^2}{E^2}}. \tag{60}
\]

Well above the threshold

\[
\sigma_{\text{tot}} \approx \sigma_{0\text{tot}}^\text{tot} \equiv \frac{4\pi}{3} \frac{\alpha^2}{s}. \tag{61}
\]

At the threshold the cross-section is zero, but it rises very rapidly with energy and for $E = 1.25M_\mu$, $\sigma_{\text{tot}}$ reaches 80% of $\sigma_{0\text{tot}}^\text{tot}$.