**Plane Electromagnetic Waves**

**Review of Waves**

A wave in one space dimension is parametrized by a function of two variables, space coordinate $x$ and time $t$. For example, a wave on a string is described by the transverse displacement $y$ as a function of $x$ and $t$. This function obeys the *wave equation*, which is a second-order partial differential equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \times \frac{\partial^2 y}{\partial t^2}. \quad (1)$$

The general solution of eq. (1) is a superposition of an arbitrary wave pulse traveling right at speed $v$ and an independent wave pulse traveling left at the same speed $v$,

$$y(x,t) = f_1(x - vt) + f_2(x + vt) \quad (2)$$

for arbitrary functions $f_1$ and $f_2$ of a single variable. Of particular interest are harmonic traveling waves of the form

$$y(x,t) = A \times \cos \left( \frac{2\pi}{\lambda} \times x - 2\pi f \times t \right) \quad \text{or} \quad y(x,t) = A \times \cos \left( \frac{2\pi}{\lambda} \times x + 2\pi f \times t \right), \quad (3)$$

where $f =$ is the cyclic frequency of the wave — so that $T = 1/f$ is its period in time — while $\lambda$ is the wave’s period in space, usually called the *wavelength*. The frequency and the wavelength are related to the speed of the wave as

$$f \times \lambda = v. \quad (4)$$

It is often to express the harmonic traveling wave in terms of the angular frequency $\omega = 2\pi f$ and wave number

$$k \overset{\text{def}}{=} \frac{2\pi}{\lambda} = \frac{\omega}{v} \quad (5)$$

as

$$y(x,t) = A \times \cos(kx - \omega t) \quad \text{or} \quad y(x,t) = A \times \cos(kx + \omega t). \quad (6)$$
A wave in 3 space dimension is parametrized by a function of four variables, the time $t$ and the 3 space coordinates $(x, y, z)$. For example, a sound wave in air is parametrized by the variance of air pressure

$$\delta P(x, y, z, t) = P(x, y, z, t) - P_{\text{avg}}$$

as a function of $(x, y, z, t)$. The \textit{3D wave equation} is also a second-order partial differential equation, but it involved derivatives WRT all 4 variables,

$$\Delta \delta P(x, y, z, t) \overset{\text{def}}{=} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \delta P(x, y, z, t) = \frac{1}{v^2} \times \frac{\partial^2}{\partial t^2} \delta P(x, y, z, t). \quad (8)$$

The 3D wave equation has an infinite number of independent solutions, and I cannot possibly describe them all here. Instead, let me focus on a particular type of solution, the plane wave pulse traveling in the $+x$ direction

$$\delta P(x, y, z, t) = f(x - vt) \quad \text{regardless of } y, z. \quad (9)$$

Such waves are called \textit{plane waves} since their wave fronts form flat planes $\perp$ to the direction of the wave’s propagation. Of particular importance are the \textit{harmonic plane waves} of the form

$$\delta P(x, y, z, t) = A \times \cos \left( \frac{2\pi}{\lambda} \times x - 2\pi f \times t \right) = A \times \cos(kx - \omega t). \quad (10)$$

More generally, a harmonic plane wave traveling in some generic direction has form

$$\delta P(x, y, x, t) = A \times \cos(k \cdot r - \omega t) = A \times \cos(k_x \times x + k_y \times y + k_z \times z - \omega \times t) \quad (11)$$

where $k$ is the \textit{wave vector} in the direction of the wave’s propagation and of magnitude

$$|k| = k = \frac{2\pi}{\lambda} = \frac{\omega}{v}. \quad (12)$$
### Wave Equation for the Electromagnetic Waves

The electric field $E$ and the magnetic field $B$ obey wave equations similar to (8). To derive such wave equations — and also to calculate the speed of EM waves — let’s start with the Maxwell equations (in MKSA units)

\[
\begin{align*}
\nabla \cdot D &= \rho, \\
\nabla \cdot B &= 0, \\
\nabla \times E &= -\frac{\partial}{\partial t} B, \\
\nabla \times H &= J + \frac{\partial}{\partial t} D, \\
\end{align*}
\]

where

\[
D = \kappa \epsilon_0 \times E \quad \text{and} \quad H = \frac{1}{\mu_{\text{rel}} \mu_0} \times B. \tag{14}
\]

Let’s assume that the electric and magnetic fields propagate in a uniform medium with constant dielectric constant $\kappa$ and constant relative permeability $\mu_{\text{rel}}$. This allows us to eliminate the $D$ and $H$ in favor of the the $E$ and $B$ fields only, so the Maxwell equations become

\[
\begin{align*}
\nabla \cdot E &= \frac{1}{\kappa \epsilon_0} \times \rho, \\
\nabla \cdot B &= 0, \\
\nabla \times E &= -\frac{\partial}{\partial t} B, \\
\nabla \times B &= \mu_{\text{rel}} \mu_0 \times J + \mu_{\text{rel}} \mu_0 \kappa \epsilon_0 \times \frac{\partial}{\partial t} E. \\
\end{align*}
\]

Let’s also assume no external electric charges or currents, $\rho \equiv 0$ and $J \equiv 0$, so we may focus on the propagation of the EM waves rather than creation of such waves in the first place.

To simplify the last of the equations (15), let us define

\[
\begin{align*}
n &= \sqrt{\kappa \times \mu_{\text{rel}}}, \quad c &= \frac{1}{\sqrt{\mu_0 \times \epsilon_0}} = 299 792 458 \text{ m/s}, \tag{16}
\end{align*}
\]

so that the messy coefficient of the last term on the RHS becomes simply

\[
\mu_{\text{rel}} \mu_0 \kappa \epsilon_0 = \frac{n^2}{c^2}. \tag{17}
\]
We shall see momentarily that its is this coefficient which determines the speed of the EM wave,

\[ v = \frac{c}{n}. \]  

(18)

In particular, in the vacuum \( n = 1 \), and the speed of the EM waves is \( c = 299792458 \) \( \text{m/s} \).

To derive the wave equations from the Maxwell equations (15), we need a vector identity relating a double cross product to a dot product,

for any 3 vectors \( a, b, c : \ a \times (b \times c) = b(a \cdot c) - (a \cdot b)c. \)  

(19)

In particular, for \( a = b = \nabla \) while \( c \) is the electric field \( E \) or the magnetic field \( B \), we have

\[ \nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E, \]  

\[ \nabla \times (\nabla \times B) = \nabla(\nabla \cdot B) - \nabla^2 B, \]  

(20)

where \( \nabla^2 = \Delta \) is the Laplacian operator

\[ \nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \]  

(21)

More generally, the curl of a curl of a vector field is the gradient of the field’s divergence minus Laplacian of the field,

\[ \text{curl(curl}(A)) = \text{grad(div}(A)) - \text{Laplacian}(A). \]  

(22)

But for the electric and magnetic field, the first term on the RHS vanishes since by Gauss Law \( \nabla \cdot B = 0 \) and \( \nabla \cdot E = 0 \) (for \( \rho \equiv 0 \)), so the curl of the field’s curls is simply minus the Laplacian,

\[ \nabla \times (\nabla \times E) = -\Delta E, \]  
\[ \nabla \times (\nabla \times B) = -\Delta B. \]  

(23)

Now let’s compare these formulae for the curl of a curl to the last two Maxwell equa-
tions (15). Let’s take the curl of both sided of the third equation (the induction equation):

\[ \nabla (\text{LHS}_3) = \nabla \times (\nabla \times E) = -\Delta E \]

\[ \langle \langle \text{by the first eq. (23)} \rangle \rangle \]

\[ \nabla (\text{RHS}_3) = \nabla \times \left( -\frac{\partial B}{\partial t} \right) \]

\[ = -\frac{\partial}{\partial t} (\nabla \times B) \]

\[ = -\frac{\partial}{\partial t} \left( \frac{n^2}{c^2} \times \frac{\partial E}{\partial t} \right) \]

\[ \langle \langle \text{by the fourth Maxwell equation} \rangle \rangle \]

\[ = -\frac{n^2}{c^2} \times \frac{\partial^2}{\partial t^2} E, \] (24)

and therefore

\[ \Delta E = +\frac{n^2}{c^2} \times \frac{\partial^2}{\partial t^2} E. \] (25)

Thus, the electric field obeys the wave equation with wave speed

\[ \frac{1}{v^2} = \frac{n^2}{c^2} \implies v = \frac{c}{n}. \] (26)

Similarly, let’s take the curl of both sides of the fourth Maxwell equation (15) (for \( J \equiv 0 \)):

\[ \nabla (\text{LHS}_4) = \nabla \times (\nabla \times B) = -\Delta B \]

\[ \langle \langle \text{by the second eq. (23)} \rangle \rangle \]

\[ \nabla (\text{RHS}_4) = \nabla \times \left( \frac{n^2}{c^2} \times \frac{\partial E}{\partial t} \right) \]

\[ = \frac{n^2}{c^2} \times \frac{\partial}{\partial t} (\nabla \times E) \]

\[ = \frac{n^2}{c^2} \frac{\partial}{\partial t} \left( -\frac{\partial B}{\partial t} \right) \]

\[ \langle \langle \text{by the third Maxwell equation} \rangle \rangle \]

\[ = -\frac{n^2}{c^2} \times \frac{\partial^2}{\partial t^2} B, \] (27)
and therefore
\[ \Delta \mathbf{B} = \frac{n^2}{c^2} \cdot \frac{\partial^2}{\partial t^2} \mathbf{B}. \] (28)

Thus, the magnetic field also obeys the wave equation with the same wave speed \( v = \frac{c}{n} \).

Note: The vacuum permittivity \( \varepsilon_0 \) and vacuum permeability \( \mu_0 \) are universal constants (in the MKSA units), so the speed of light in the vacuum
\[ c = \frac{1}{\sqrt{\mu_0 \times \varepsilon_0}} = 299,792,458 \text{ m/s}, \] (29)
is a universal constant. Thus, in the vacuum, the EM waves with all frequencies — from milli-Hertz to \( 10^{25} \) Hz — move at exactly the same speed \( c \).

But in a dielectric or a magnetic medium, the dielectric constant \( \kappa \) and/or relative permeability \( \mu_{\text{rel}} \) may depend on the wave’s frequency, so \( n = \sqrt{\kappa \mu_{\text{rel}}} \) and hence wave speed \( c/n \) may change with frequency. This is known as dispersion. For example, at low frequencies, water has a very large dielectric constant \( \kappa \approx 80 \), which leads to \( n \approx 9 \) and hence rather slow wave speed \( v = 0.11 \, c \). But at frequencies of the visible light waves, water’s dielectric constant drops to \( \kappa \approx 1.78 \), hence \( n \approx \frac{4}{3} \) and much faster wave speed \( v \approx 0.75 \, c \).

**Plain Electromagnetic Waves**

To learn more about the electromagnetic waves, let’s consider a plane harmonic EM wave in some detail. By plane harmonic wave I mean that both the electric field and the magnetic field depend on position and time according to
\[ \mathbf{E}(\mathbf{r}, t) = E_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t), \quad \mathbf{B}(\mathbf{r}, t) = B_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta \phi). \] (30)*

Both the electric and the magnetic waves here have the same wave vector \( \mathbf{k} \) and frequency \( \omega = \nu \times |\mathbf{k}| \), but different amplitude vectors \( \mathbf{E}_0 \) and \( \mathbf{B}_0 \). In principle, there could be also a phase difference \( \delta \phi \) between the two waves but we shall see in a moment that \( \delta \phi = 0 \). We

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* In this section of the notes I would like to reserve the symbols ‘\cdot’ and ‘\times’ for the dot-product and cross-product of two vectors, so when I need to emphasise a product of a scalar and a vector or of two scalar, I use ‘∗’.
shall also see that the electric and the magnetic amplitudes $E_0$ and $B_0$ are related to each other.

Indeed, let’s plug in the fields (30) into the Maxwell equations, starting with the Gauss Laws

$$\nabla \cdot E = \frac{1}{\kappa \epsilon_0} \times \rho = 0 \quad \text{and} \quad \nabla \cdot B = 0. \quad (31)$$

For the electric field as in eq. (30), with a constant amplitude $E_0$, we have

$$\nabla \left( E = \cos(k \cdot r - \omega t) \ast E_e \right) = \left( \nabla \cos(k \cdot r - \omega t) \right) \cdot E_0 \quad (32)$$

where

$$\nabla \cos(k \cdot r - \omega t) = \frac{d \cos(k \cdot r - \omega t)}{d(k \cdot r - \omega t)} \ast k = -\sin(k \cdot r - \omega t) \ast k,$$

hence

$$\nabla \cdot E = -\sin(k \cdot r - \omega t) \ast (E_0 \cdot k). \quad (33)$$

Since the Gauss Law requires $\nabla \cdot E = 0$ everywhere and everywhen, the coefficient of the $\sin(\cdots)$ in this formula must vanish, thus $E_0 \cdot k = 0$ — the amplitude vector $E_0$ of the electric field must be $\perp$ to the wave vector $k$.

Likewise, for the magnetic field as in (30), we have

$$\nabla \cdot \left( B = \cos(k \cdot r - \omega t + \delta \phi) \ast B_0 \right) = -\sin(k \cdot r - \omega t + \delta \phi) \ast (B_0 \cdot k), \quad (34)$$

so the Gauss Law $\nabla \cdot B = 0$ (everywhere) requires $B_0 \cdot k = 0$ — the magnetic amplitude vector should also be $\perp$ to the wave vector $k$. Thus, the EM waves are transverse — both the electric and the magnetic fields of the wave are always perpendicular to the direction $k$ of the wave’s propagation.
Next, let’s work out the induction equation

\[ \nabla \times E = -\frac{\partial}{\partial t} B. \quad (35) \]

For the fields as in eq. (30), we have

\[ \nabla \times \left( E = \cos(k \cdot r - \omega t) * E_0 \right) = \nabla \left( \cos(k \cdot r - \omega t) \right) \times E_0 = -\sin(k \cdot r - \omega t) * (k \times E_0) \quad (36) \]

while

\[ \frac{\partial}{\partial t} \left( B = \cos(k \cdot r - \omega t + \delta \phi) * B_0 \right) = \omega \sin(k \cdot r - \omega t + \delta \phi) * B_0. \quad (37) \]

Plugging these derivatives into the induction equation (35), we obtain

\[ -\sin(k \cdot r - \omega t) * (k \times E_0) = -\omega \sin(k \cdot r - \omega t + \delta \phi) * B_0 \quad (38) \]

which should hold true everywhere and everywhen. Consequently, we must have \( \delta \phi = 0 \) — otherwise, the maxima and the minima of the two waves would not match — so the electric and the magnetic waves have the same phase. We also need equal coefficients of the sine wave, thus

\[ k \times E_0 = \omega * B_0. \quad (39) \]

This vector relation implies that the \( E_0 \) and \( B_0 \) vector amplitudes are perpendicular to each other. Therefore, in a plain EM wave, the electric and the magnetic fields are always perpendicular to each other as well as to the direction of the wave’s propagation. For example, if the wave propagates in the +x direction and the electric field is parallel to the y axis than the magnetic field is parallel to the z axis.
Besides directions, eq. (39) relates the magnitudes of the electric and magnetic amplitudes:

$$\omega * |B_0| = |k \times E_0| = |k| * |E_0| \quad \langle \text{since } k \perp E_0 \rangle$$

(40)

and therefore

$$|B_0| = \frac{|k|}{\omega} \times |E_0| = \frac{1}{v} \times |E_0|. \quad (41)$$

We shall see in a moment that the Maxwell–Ampere equation establishes a similar relation between the electric and magnetic amplitudes, but with a different dependence of the wave speed $v$, and making sure the two relations are consistent with each other determines the wave speed to be $v = c/n$.

Indeed, let’s work out the Ampere–Maxwell equation

$$\nabla \times B = + \frac{n^2}{c^2} \frac{\partial}{\partial t} E \quad (42)$$

(in the absence of electric current $J$). For the fields as in eq. (30),

$$\nabla \times \left( B = \cos(k \cdot r - \omega t + \delta \phi) \times B_0 \right) = - \sin(k \cdot r - \omega t + \delta \phi) \times (k \times B_0), \quad (43)$$

$$\frac{\partial}{\partial t} \left( E = \cos(k \cdot r - \omega t) \times E_0 \right) = + \omega \sin(k \cdot r - \omega t) \times E_0, \quad (44)$$

so the Ampere–Maxwell equation becomes

$$- \sin(k \cdot r - \omega t + \delta \phi) \times (k \times B_0) = + \frac{n^2}{c^2} \omega \times \sin(k \cdot r - \omega t) \times E_0. \quad (45)$$

Again, this equation must hold everywhere and everywhen, which requires $\delta \phi = 0$ as well as

$$k \times B_0 = - \frac{n^2}{c^2} \omega \times E_0. \quad (46)$$

This vector relation requires the amplitude vectors $E_0$ and $B_0$ to be $\perp$ to each other — which we have already established from the induction equation — and also relates the magnitudes
of the two amplitudes as

$$|B_0| = \frac{n^2}{c^2} \ast \frac{\omega}{|k|} \ast |E_0| = \frac{n^2}{c^2} \ast v \ast |E_0|.$$  (47)

Comparing eqs. (41) and (47) and demanding that both relations hold true at the same time, we obtain

$$\frac{|B_0|}{|E_0|} = \frac{1}{v} = \frac{n^2}{c^2} \ast v$$  (48)

hence

$$\frac{1}{v^2} = \frac{n^2}{c^2}$$  (49)

and therefore

$$v = \frac{c}{n}.$$  (50)

In particular, an EM wave in the vacuum (where \(n = 1\)) always has speed \(v = c\).

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To summarize, (1) in a plain harmonic EM wave, the electric and the magnetic field are always perpendicular to each other as well as to the direction of the wave’s propagation. (2) The speed of any EM wave in the vacuum is

$$v = c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 299 792 458 \text{ m/s},$$  (51)

while in a dielectric and/or magnetic medium the EM waves have a slower speed

$$v = \frac{c}{n} \text{ where } n = \sqrt{\mu_{\text{rel}} \kappa} > 1.$$  (52)

(3) Plane harmonic EM waves have form

$$E(r, t) = \cos(k \cdot r - \omega t) \ast E_0, \quad B(r, t) = \cos(k \cdot r - \omega t) \ast B_0,$$  (53)

where the wave vector \(k\) has magnitude \(k = \omega/v = (n/c) \omega\) and points in the direction of the wave’s propagation. (4) The amplitude vectors of the electric and magnetic fields are
related to each other as

\[ \mathbf{B}_0 = \frac{\mathbf{v}}{v^2} \times \mathbf{E}_0. \]  \hfill (54)

(5) This relation leads to the right hand rule for the directions of the fields: Let the index finger of your right hand point in the direction of the wave’s velocity \( \mathbf{v} \). Then, if your bend middle finger points in the direction of \( \mathbf{E} \) than your thumb points in the direction of \( \mathbf{B} \). For example,

Note that the \( \mathbf{E} \) and \( \mathbf{B} \) fields change their directions every half-wavelength, but they do it at the same time in accordance with the right hand rule.

(6) Finally, the magnitudes of the electric and magnetic fields in an EM wave are related to each other as

\[ |\mathbf{B}_0| = \frac{|\mathbf{E}_0|}{v}. \]  \hfill (55)

For example, if the EM wave in the vacuum has electric amplitude \(|\mathbf{E}_0| = 1200 \text{ V/m} \), then the magnetic amplitude is

\[ |\mathbf{B}_0| = \frac{1200 \text{ V/m}}{3.0 \cdot 10^8 \text{ m/s}} = 4.0 \cdot 10^{-6} \text{ T}. \]  \hfill (56)

But a wave of the same electric amplitude in a glass with \( n = 1.5 \) will have a larger magnetic amplitude

\[ |\mathbf{B}_0| = \frac{|\mathbf{E}_0|}{c/n} = \frac{1200 \text{ V/m}}{(3.0 \cdot 10^8 \text{ m/s})/1.5} = 6.0 \cdot 10^{-6} \text{ T}. \]  \hfill (57)