BIOT–SAVART–LAPLACE LAW

Sometimes, a symmetry allows you to obtain the electric field of some charges from the Gauss Law or the magnetic field of some currents from the Ampere’s Law. But most times, the symmetry is not there, so it’s time to shut up and integrate... In the electric case, we integrate the Coulomb formula over the electric charges. In the magnetic case, we integrate over the current-carrying wires and the integrand is a more complicated expression derived by Jean Baptiste Biot, Félix Savart, and Pierre-Simon Laplace.

Specifically, the magnetic field \( \mathbf{B}(\mathbf{r}) \) due to a wire carrying current \( I \) along some line \( W \) in space — straight or curved — is given by

\[
\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \times I \times \int_W \frac{d\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}.
\]  

(1)

The integral here is a 1D line integral over the line \( W \) of the wire, which we follow in the direction of the electric current \( I \). In my notations, \( \mathbf{r} \) is the point where we measure the magnetic field \( \mathbf{B}(\mathbf{r}) \), while \( \mathbf{r}' \) is the point on a wire over which we integrate, and \( d\mathbf{r}' \) is the infinitesimal line element along the wire \( W \).

The magnetic field due to multiple wires follows from the Biot–Savart–Laplace formula (1) via the superposition principle:

\[
\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \times \sum_i^{\text{wires}} I_i \times \int_{W_i} \frac{d\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}.
\]  

(2)

Finally, for thick conductors and continuous distributions of the electric current density \( \mathbf{J}(x, y, z) \), we combine the 1D integral (1) along the wire(s) with a 2D integral \( \mathbf{J} \cdot d^2 \mathbf{A} \) across the wires into a 3D volume integral:

\[
\int_W I(\mathbf{r}') \, d\mathbf{r}' \rightarrow \int_W \left[ \int \int_{D_{\mathbf{r}'}} \mathbf{J}(\mathbf{r}') \cdot d^2 \mathbf{A}_\perp \right] \, d\mathbf{r}'_\parallel \rightarrow \iiint_{\text{whole space}} d^3 \mathbf{r}' \, \mathbf{J}(\mathbf{r}'),
\]  

(3)

In this case, the magnetic field becomes

\[
\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint_{\text{whole space}} d^3 \mathbf{r}' \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}.
\]  

(4)
Mathematically, the Biot–Savart–Laplace formula (4) is the solution of partial differential equations

\[ \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad (5) \]

which stem from the Gauss’s and Ampere’s Lawa for the magnetic field. In these notes, I shall not try to derive this solution or even prove that it is a solution; the proof will have to wait for the upper-level the Electrodynamics class. Instead, I shall give a few example of using the Biot–Savart–Laplace formula (1). For simplicity, I’ll stick to a single thin wire, so you would not need the volume integral (4).

**Example: Circular Wire**
Let’s start with the example of a wire shaped into a circle of radius \( R \):

For simplicity, let me limit the calculation of the magnetic field to the axis of the ring, \( \mathbf{r} = (0, 0, z) \) only. Along the wire \( \mathbf{r}' = (R \cos \phi, R \sin \phi, 0) \), hence

\[ d\mathbf{r}' = R d\phi \times (- \sin \phi, + \cos \phi, 0) \quad (6) \]

while

\[ \mathbf{r} - \mathbf{r}' = (-R \cos \phi, -R \sin \phi, +z). \quad (7) \]
This, in the numerator of the Biot–Savart–Laplace integrand we have

\[ \mathbf{dr'} \times (\mathbf{r'} - \mathbf{r}) = \mathbf{R} d\phi (-\sin \phi, +\cos \phi, 0) \times (-\mathbf{R} \cos \phi, -\mathbf{R} \sin \phi, \mathbf{z}) \]

\[ = (z \cos \phi, z \sin \phi, \mathbf{R}) \mathbf{R} d\phi, \]  

(8)

while in the denominator

\[ |\mathbf{r} - \mathbf{r'}|^2 = \mathbf{R}^2 + \mathbf{z}^2 \implies |\mathbf{r} - \mathbf{r'}|^3 = (\mathbf{R}^2 + \mathbf{z}^2)^{3/2}. \]  

(9)

Combining the numerator and the denominator, we obtain

\[ \mathbf{B}(0, 0, z) = \frac{\mu_0}{4\pi} \times I \times \oint_W \frac{\mathbf{dr'} \times (\mathbf{r'} - \mathbf{r})}{|\mathbf{r} - \mathbf{r'}|^3} \]

\[ = \frac{\mu_0 I}{4\pi} \times \oint_W \frac{(z \cos \phi, z \sin \phi, \mathbf{R}) \mathbf{R} d\phi}{(\mathbf{R}^2 + \mathbf{z}^2)^{3/2}} \]

\[ = \frac{\mu_0 I}{4\pi} \times \frac{\mathbf{R}}{(\mathbf{R}^2 + \mathbf{z}^2)^{3/2}} \times \int_0^{2\pi} (z \cos \phi, z \sin \phi, \mathbf{R}) d\phi, \]

(10)

where in the remaining integral

\[ \int_0^{2\pi} z \cos \phi \, d\phi = 0, \]

\[ \int_0^{2\pi} z \sin \phi \, d\phi = 0, \]

(11)

\[ \int_0^{2\pi} \mathbf{R} \, d\phi = 2\pi \mathbf{R}, \]

and hence

\[ \int_0^{2\pi} (z \cos \phi, z \sin \phi, \mathbf{R}) \, d\phi = 2\pi \mathbf{R} \times \mathbf{\hat{z}}. \]  

(12)
Altogether,
\[
\mathbf{B}(0, 0, z) = \frac{\mu_0 I}{4\pi} \times \frac{R}{(R^2 + z^2)^{3/2}} \times 2\pi R \mathbf{\hat{z}} = \frac{\mu_0 I}{2R} \times \left(\frac{R^2}{R^2 + z^2}\right)^{3/2} \times \mathbf{\hat{z}}. \tag{13}
\]

Note the direction of this magnetic field: along the positive \( z \) axis.

**Example: Long Straight Wire**

For my next example I use an infinitely long wire, which I take to lie along the \( z \) axis. In this case, we already know the magnetic field of this wire, but let’s re-derive it from the Biot–Savart–Laplace formula.

In this example, the observer is at arbitrary \( \mathbf{r} = (x, y, z) \) while the wire points are \( \mathbf{r}' = (0, 0, z') \). Consequently, the BSL numerator is
\[
(0, 0, dz') \times (x, y, z - z') = (-y, +x, 0) \, dz'
\tag{14}
\]
while the BSL denominator is
\[
(x^2 + y^2 + (z - z')^2)^{3/2},
\tag{15}
\]
so altogether
\[
\mathbf{B}(x, y, z) = \frac{\mu_0 I}{4\pi} \times \int_{-\infty}^{+\infty} \frac{(-y, +x, 0) \, dz'}{(x^2 + y^2 + (z - z')^2)^{3/2}}
\tag{16}
\]
\[
= \frac{\mu_0 I}{4\pi} \times (-y, +x, 0) \times \int_{-\infty}^{+\infty} \frac{dz'}{(x^2 + y^2 + (z - z')^2)^{3/2}}.
\]
The remaining integral here evaluates to
\[
\int_{-\infty}^{+\infty} \frac{dz'}{(x^2 + y^2 + (z - z')^2)^{3/2}} = \frac{2}{x^2 + y^2}; \tag{17}
\]
indeed, we have already seen this integral in the context of electric field of a long thin rod.
Plugging this integral into eq. (16) gives us the magnetic field

\[
\mathbf{B}(x, y, z) = \frac{\mu_0 \times I}{2\pi} \times \frac{(-y, +x, 0)}{x^2 + y^2}.
\]  

(18)

In terms of cylindrical coordinates \((\rho, \phi, z)\), the \((-y, +x, 0)\) vector is \(\rho \hat{\phi}\), hence

\[
\frac{(-y, +x, 0)}{x^2 + y^2} = \frac{\hat{\phi}}{\rho}
\]

(19)

so the magnetic field is

\[
\mathbf{B}(\rho, \phi, z) = \frac{\mu_0 \times I}{2\pi \rho} \hat{\phi}.
\]

(20)

Segments:

In many cases, a wire is made of several segments. Each segment has a simple geometric shape — a piece of a straight line, or a circular arc — but the overall geometry can be quite elaborate. For example, consider a star made of 5 straight-line segments,

For a wire like this, the Biot–Savart–Laplace integral over the whole wire \(W\) becomes a sum of integrals over the individual segments,

\[
\mathbf{B}(\mathbf{r}) = \frac{\mu_0 \times I}{4\pi} \times \sum_{\text{segments}} \int_{\text{segment } \# i} \frac{d\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}.
\]

(21)

Let’s work out this integrals here for the straight-line and the circular-arc segments, and then we shall we a few interesting combinations.
**Straight-Line Wire Segment**: beginning at point \( \mathbf{r}_1' \) and ending at point \( \mathbf{r}_2' \). Let’s picture a triangle made by the two ends of this segment and by the point \( \mathbf{r} \) where we measure the magnetic field:

![Diagram of straight line wire segment](image)

Since the wire segment is straight, the infinitesimal vector \( d\mathbf{\ell} = d\mathbf{r}' \) along the segment has a fixed direction, same as \( \mathbf{r}_2' - \mathbf{r}_1' \). Consequently, the vector product in the numerator of the BSL integral remains constant along the whole segment,

\[
d\mathbf{r}' \times (\mathbf{r} - \mathbf{r}') = d\mathbf{r}' \times (\mathbf{r} - \mathbf{r}_1') - d\mathbf{r}' \times (\mathbf{r}_2' - \mathbf{r}_1') \\
\equiv d\mathbf{r}' \times (\mathbf{r} - \mathbf{r}_1') \\
= d\mathbf{\ell} \times \mathbf{h},
\]

where \( d\mathbf{\ell} = d\mathbf{r}' \) is the infinitesimal length element along the straight segment, and \( \mathbf{h} \) is the height if the triangle (22). In other words, \( \mathbf{h} \) is the line from the point \( \mathbf{r} \) where we measure the magnetic field to the wire segment — or to the extrapolated straight line of the wire segment — in the direction \( \perp \) to the segment.

Note: if we measure the magnetic field at a point \( \mathbf{r} \) which happens to lie right on the extrapolated straight line of the wire segment, then \( \mathbf{h} = 0 \) and hence \( d\mathbf{\ell} \times \mathbf{h} \equiv 0 \). Consequently, the whole BSL integral vanishes regardless of the denominator’s details, and the magnetic field of the segment is zero. Thus, **straight segments ‘pointing’ directly towards or directly away from \( \mathbf{r} \) do not contribute to the magnetic field at \( \mathbf{r} \).**
For $\vec{h} \neq 0$, the direction of the magnetic field is the direction of the vector product $d\ell \times \vec{h}$ in the numerator of the BSL integral. This direction is $\perp$ to the wire and to the $\vec{h}$; in other words, the direction of $\mathbf{B}(\mathbf{r})$ is $\perp$ to the whole triangle (22). The specific perpendicular obtains from the right screw rule: If from your point of view, the current flows in the clockwise direction around $\mathbf{r}$ — as it does on figure (22) — then take the perpendicular which points away from you. OOH, if you see the current flows counterclockwise around $\mathbf{r}$, then take the perpendicular which points towards you.

Now that we know the direction of the magnetic field, let’s find its magnitude

$$B = \frac{\mu_0 \times I}{4\pi} \times \frac{\int_{\ell_1}^{\ell_2} d\ell \times \vec{h}}{|\mathbf{r}' - \mathbf{r}|^3}$$

(24)

In this formula, $\ell$ is the coordinate along the wire, and I take its origin $\ell = 0$ to be the point $O$ where the height $\vec{h}$ of the triangle touches the wire or the extrapolated line of the wire. In terms of this $\ell$,

$$|\mathbf{r}' - \mathbf{r}|^2 = \ell^2 + h^2 \implies |\mathbf{r}' - \mathbf{r}|^3 = \left(\ell^2 + h^2\right)^{3/2},$$

(25)

so the BSL integral (24) becomes

$$B = \frac{\mu_0 \times I}{4\pi} \times \frac{\int_{\ell_1}^{\ell_2} h \times d\ell}{\left(\ell^2 + h^2\right)^{3/2}}.$$  

(26)

To evaluate this integral, we change the integration variable from $\ell$ to the angle

$$\alpha = 90^\circ + \arctan \frac{\ell}{h} = \arccot \frac{h}{\ell}.$$  

(27)

Note: the angles $\alpha_1$ and $\alpha_2$ on figure (22) are precisely the $\alpha$’s corresponding to the ends of
the wire segment. In terms of $\alpha$,

$$\ell = -h \times \cot \alpha, \quad d\ell = \frac{h \, d\alpha}{\sin^2 \alpha}, \quad \ell^2 + h^2 = \frac{h^2}{\sin^2 \alpha}, \quad (28)$$

hence

$$\frac{h \times d\ell}{(\ell^2 + h^2)^{3/2}} = h \times \frac{h \, d\alpha}{\sin^2 \alpha} \times \frac{\sin^3 \alpha}{h^3} = \frac{\sin \alpha}{h} \, d\alpha = -\frac{d \cos \alpha}{h},$$

and therefore

$$\int_{\ell_1}^{\ell_2} \frac{h \times d\ell}{(\ell^2 + h^2)^{3/2}} = \frac{\cos \alpha_2}{h} - \frac{\cos \alpha_1}{h}, \quad (29)$$

Altogether, the magnetic field of a straight wire segment is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 \times I}{4\pi h} \times (\cos \alpha_1 - \cos \alpha_2) \times \mathbf{i}_\perp \triangle \quad (30)$$

where $h$ and the angles $\alpha_1$ and $\alpha_2$ are as shown on figure (22) and $\mathbf{i}_\perp \triangle$ is the unit vector $\perp$ to the whole triangle.

Note: in the limit of infinitely long segment in both directions, $\alpha_1 \to 0, \alpha_2 \to \pi$, hence $\cos \alpha_1 - \cos \alpha_2 \to 2$, and the magnetic field of the segment agrees with the formula for an infinite wire,

$$B_{\infty} = \frac{\mu_0 \times I}{2\pi h}. \quad (31)$$

**Example: Magnetic Field of a Square Loop**

Consider a closed loop of wire in the shape of an $a \times a$ square:

Let’s calculate the magnetic field at the center of the square (shown in blue).
The square wire consists of 4 similar straight-line segments, so all we need is to evaluate eq. (30) for the magnetic field due to each segment, and then total up the 4 segments’ contributions. For each segment, \( h = \frac{1}{2}a, \alpha_1 = 45^\circ, \alpha_2 = 135^\circ \), hence

\[
B_{1 \text{ segment}} = \frac{\mu_0 \times I}{2\pi a} \times (\cos 45^\circ - \cos 135^\circ = \sqrt{2}) = \frac{\sqrt{2}}{2\pi} \times \frac{\mu_0 I}{a}.
\]

Also, for each segment the triangle spanning the wire and the center of the square where we measure \( B \) lie in the plane of the square, so the direction of the magnetic field due to each segment is \( \perp \) to the whole square. Specifically, the magnetic field points into the screen (where you are reading this) since in each segment the current flows clockwise around the center. Thus, altogether, the magnetic field points into the screen and its magnitude is

\[
B_{\text{whole square}} = 4 \times B_{1 \text{ segment}} = \frac{4\sqrt{2}}{2\pi} \times \frac{\mu_0 I}{a} = \frac{8\sqrt{2}}{\pi} \times \frac{\mu_0 I}{4a}.
\]  

**Example: Symmetric \( N \)-sided Polygon**

In this example the wire also makes a complete loop, this time in the shape of symmetric \( N \)-sided polygon with side \( a \), for example

![Hexagon](image)

Again, we focus on the magnetic field at the center of the polygon, so by symmetry each segment of the wire contributes a similar \( B_{1 \text{ segment}} \). All these contributions are directed \( \perp \) to the polygon, specifically into the screen, hence

\[
\mathbf{B}_{\text{polygon}} = N \times B_{1 \text{ segment}} \times \mathbf{n}_{\text{into screen}}.
\]

Now let’s draw a single segment of the wire and the triangle connecting it to the center point
Simple geometry + trigonometry for this triangle gives us

\[ \begin{align*}
\alpha_{1,2} &= \frac{\pi}{2} \mp \frac{\pi}{N}, \\
\cos \alpha_1 - \cos \alpha_2 &= 2 \sin \frac{\pi}{N}, \\
h &= \frac{a}{2} \times \cot \frac{\pi}{N},
\end{align*} \tag{33, 34, 35} \]

and therefore

\[ B_{1 \text{ segment}} = \frac{\mu_0 I}{4\pi} \times \frac{\cos \alpha_1 - \cos \alpha_2}{h} = \frac{\mu_0 I}{4\pi} \times \frac{2 \tan \frac{\pi}{N}}{a} \times 2 \sin \frac{\pi}{N}. \tag{36} \]

Finally, combining all \( N \) segments, we find the magnetic field at the center of the polygon is

\[ B = N \times B_{1 \text{ segment}} = N \times \frac{\mu_0 \times I}{\pi a} \times \sin \frac{\pi}{N} \times \tan \frac{\pi}{N} = \frac{\mu_0 \times I}{P} \times \frac{N^2}{\pi} \times \sin \frac{\pi}{N} \times \tan \frac{\pi}{N}, \tag{37} \]

where in the last expression \( P = N \times a \) is the polygon’s perimeter.

To check this formula, we first plug compare it for \( N = 4 \) with eq. (32) for the square and see that they indeed produce the same magnetic field at the center. Second, let’s take a large \( N \) limit in which the polygon becomes a circular ring of perimeter \( P = 2\pi R \). In this
limit,
\[
\lim_{N \to \infty} \left( \frac{N^2}{\pi} \times \sin \frac{\pi}{N} \times \tan \frac{\pi}{N} \right) = \pi,
\]
(38)
hence the magnetic field at the center of the polygon becomes
\[
B = \frac{\mu_0 \times I}{2\pi R} \times \pi = \frac{\mu_0 \times I}{2},
\]
(39)
which is indeed the field at the center of a circular ring.

**Example: Magnetic Field of a Circular Arc**

As our final example, let’s calculate the magnetic field at the center of a circular arc. More generally, consider a wire comprised of a semicircle and two straight segments

![Diagram of a circular arc with point C at the center](image)

and calculate the magnetic field at point C at the center of the circular arc.

Note: besides being at the center of the arc, the point C happens to lie on the straight-line extrapolations of the straight segments of the wire. Consequently, the two straight segments do not contribute to the magnetic field \( B(C) \) at that point. Thus, the entire field at point C comes from the circular arc segment only.

Let’s parametrize the arc segment by angle \( \phi \) from the point C; \( \phi \) ranges from 0 to \( \varphi \). In terms of \( \phi \),

\[
\mathbf{r}' = (R \cos \phi, R \sin \phi, 0),
\]
(40)
\[
\mathbf{r} - \mathbf{r}' = (-R \cos \phi, -R \sin \phi, 0),
\]
(41)
\[
d\mathbf{r}' = (-R \sin \phi, +R \cos \phi, 0) \, d\phi,
\]
(42)
hence in the numerator of the Biot–Savart–Laplace integral
\[ dr' \times (r - r') = (-R \sin \phi, +R \cos \phi, 0) \, d\phi \times (-R \cos \phi, -R \sin \phi, 0) = (0, 0, +R^2 \, d\phi). \]

Note: the direction of this vector product is always vertically Up, \( \perp \) to the plane of the ring, so the magnetic field’s direction is going to be vertically Up.

As to the denominator of the BSL formula, the whole circular arc is at constant distance \(|r - r'| \equiv R\) from the ring’s center, so the denominator is a constant \(R^3\). Altogether,
\[
\int_{\text{arc}} \frac{dr' \times (r - r')}{|r - r'|^3} = \int_{0}^{\varphi} \frac{R^2 (0, 0, 1) \, d\phi}{R^3} = \frac{\varphi}{R} \times (0, 0, 1),
\]
(43)
so the magnetic field at point C is
\[
\mathbf{B}(C) = \frac{\mu_0 \times I \times \varphi}{4\pi R} \times \hat{z}.
\]
(44)
Note: the \( \varphi \) angle in this formula should be taken in radians.