Mathematical Aspects of the Magnetic Field

Faraday Law of Magnetic Induction

The Faraday’s Law of magnetic induction is fairly simple. Consider a loop of wire, or more generally a closed electric circuit laid down along some closed line $\mathcal{L}$ in space. The magnetic flux $\Phi_B$ through the loop (or contour) is basically the number of magnetic field lines which are surrounded by loop. If this flux changes for any reason whatsoever, the change induces the electromotive force in the contour

$$\mathcal{E} = -\frac{d}{dt}\Phi_B[\mathcal{L}],$$

(1)

where the minus sign implements the Lenz rule. It does not matter if the flux changes because of changing magnetic field $\mathbf{B}(t)$ or because the wire making the loop $\mathcal{L}$ moves in space, the only thing which matter for the Faraday’s Induction Law is the magnetic flux through the loop and its overall change with time.

But in the differential form of the Induction Law, the time-dependent magnetic field $\mathbf{B}(\mathbf{r}; t)$ induces an electric field and hence EMF in a way which seems completely unrelated to the motional EMF induced in a moving wire loop in a constant magnetic field. Nevertheless, the two ways of induction complement each other, so on the bottom line it does not matter if we move a wire loop around a stationary magnet or move a magnet around a stationary wire loop, the net EMF is the same in both cases. This complementarity shows that in electromagnetism, the motion is just as relative as it is in classical mechanics. Indeed, Albert Einstein derived his Special Relativity theory from this very observation!

Alas, the Special Relativity is beyond the scope of this class. Instead, in these notes I will show you the differential Maxwell’s Induction Equation and explain how this equation together with the Lorentz force give rise to the Faraday’s Law of Induction.

In electrostatics, the electric field $\mathbf{E}(x, y, z)$ gives rise to a conservative force, so the integral $\oint \mathbf{E} \cdot d\mathbf{l}$ over any closed loop must vanish. In a differential form, this means that the electric field has zero curl, $\nabla \times \mathbf{E} \equiv 0$, and the electric field itself is (minus) the gradient of a scalar potential, $\mathbf{E} = -\nabla V(x, y, z)$. But in electrodynamics, none of these statements holds
true when the magnetic field changes with time! Instead, a time dependent magnetic field induces a non-potential electric field with a non-zero curl according to Maxwell’s Induction Equation

\[
\nabla \times E(x, y, x; t) = -\frac{\partial}{\partial t} B(x, y, z; t)
\]

(2)

(in MKSA units). For example, a spatially uniform (but time-dependent) magnetic field in \( z \) direction \( B = B(t) \hat{z} \) induces circular electric field in the \((x, y)\) plane,

\[
E = -\frac{1}{2} \frac{\partial B}{\partial t} \times \rho \hat{\phi}
\]

\[
= \frac{1}{2} \frac{\partial B}{\partial t} \times (+y, -x, 0).
\]

(3)

In another example, consider the cylindrical iron core with large \( \mu_{\text{rel}} \) of a long solenoid. Inside the core the magnetic field is approximately uniform, but at the core’s edge it abruptly decreases by a factor \( 1/\mu_{\text{rel}} \), and outside the solenoid the field vanishes altogether. To a good approximation, \( B = B(t) \hat{z} \) inside the core but outside the core \( B = 0 \). If the current through the solenoid and hence \( B(t) \) in the core change with time, solving eq. (2) for the electric field gives
\[ \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \times \mathbf{i} z \times \begin{cases} \frac{\rho}{2} & \text{for } \rho < R, \\ \frac{R^2}{2\rho} & \text{for } \rho > R. \end{cases} \]  

In both examples, we have circular electric field which cases EMF in a circular wire loop.

In general, the EMF in a fixed wire loop due to a time-dependent electric field is caused by the non-potential electric field (2) induced by the magnetic field. On the other hand, the motional EMF induced in a moving wire loop arises from the Lorentz force

\[ \mathbf{F}_L = q \mathbf{v} \times \mathbf{B} \]  

on the electrons in the moving wires. Or in a circuit loop made from more exotic moving conductors — semiconductors, electrolytes, plasma, whatever — the motional EMF arises from the Lorentz force on the available charge carries, be they electrons, holes, ions, or whatever.

To see how the induced electric field and/or the Lorentz force give rise to EMF, consider the net force

\[ \mathbf{F}_{\text{net}} = q (\mathbf{E}_{\text{induced}} + \mathbf{v} \times \mathbf{B}). \]  

on a generic current-carrying particle of charge \( q \). This force has no reason to be potential, so when a charged particle moves in a complete circuit around the loop \( \mathcal{L} \) — or rather, when a whole bunch of current-carrying particles move along the loop so that the net charge \( Q \)
makes a complete circuit — the force (6) performs non-zero work

\[ W = Q \times \oint_L (\mathbf{E}_{\text{induced}} + \mathbf{v} \times \mathbf{B}) \cdot d\vec{l}. \]  

(7)

By definition of the electromotive force, the EMF is the work supplied by outside energies — mechanical, chemical, electric, whatever — to propel a unit of charge around the electric circuit. Therefore, the net EMF induced by changing the magnetic field and/or moving the loop in that magnetic field amounts to

\[ \mathcal{E} = \oint_L (\mathbf{E}_{\text{induced}} + \mathbf{v} \times \mathbf{B}) \cdot d\vec{l}. \]  

(8)

We shall see momentarily that the induction equation (2) leads to the integral identity

\[ \oint_L (\mathbf{E}_{\text{induced}} + \mathbf{v} \times \mathbf{B}) \cdot d\vec{l} = -\frac{d}{dt} \Phi_B[\text{through } L] \]  

(9)

which recasts the EMF (8) in terms of the Faraday’s Law

\[ \mathcal{E} = -\frac{d}{dt} \Phi_B[\text{through } L]. \]  

(1)

Thus, the Faraday’s Law comprises both the induction equation (2) and the electromotive force (8) stemming from electric and Lorentz forces.

\[ \star \quad \star \quad \star \]

The rest of these notes explain the math behind the relation (9). Let’s start by clarifying the precise meaning of the magnetic flux through a closed loop \( \mathcal{L} \). We know how to define the flux through an orientable surface \( \mathcal{S} \),

\[ \Phi_B[\mathcal{S}] = \iint_{\mathcal{S}} \mathbf{B}(\mathbf{r}) \cdot d^2\mathbf{A} \]  

(10)

so to define the flux through a loop \( \mathcal{L} \), we need a surface \( \mathcal{S} \) which spans \( \mathcal{L} \). That is, \( \mathcal{S} \) has topology of a disk with \( \mathcal{L} \) as its boundary. But there are many such surfaces, so which should we choose?
Fortunately, thanks to the Gauss Law, all surfaces spanning the same loop $\mathcal{L}$ have the same magnetic flux going through them. Indeed, consider two surfaces $S_1$ and $S_2$ spanning $\mathcal{L}$. Let’s invert the orientation of the $S_2$ and then glue it to the $S_1$ along their common boundary $\mathcal{L}$. The combined surface $S_1 - S_2$ has no boundary, and all such boundary-less surfaces are themselves complete boundaries of some volume $\mathcal{V}$. By the Gauss Law, the net magnetic flux through such complete boundary must vanish, thus

$$0 = \int\int_{S_1 - S_2} \mathbf{B} \cdot d^2 \mathbf{A} = \int\int_{S_1} \mathbf{B} \cdot d\mathbf{A} - \int\int_{S_2} \mathbf{B} \cdot d\mathbf{A} = \Phi_B[S_1] - \Phi_B[S_2]$$  

(11)

Thus, any surface spanning the same loop $\mathcal{L}$ has the same magnetic flux, so we may use any one of such surfaces to define the flux through the loop,

$$\Phi_B[\text{through } \mathcal{L}] = \Phi_B[\text{any } S \text{ spanning } \mathcal{L}].$$  

(12)

Now consider a moving wire loop $\mathcal{L}(t)$. Let’s focus on the loop geometry at some arbitrary time $t$ and at infinitesimally later time $t + \delta t$. To define the magnetic flux at time $t$ we need a surface $S$ which spans $\mathcal{L}(t)$. And we also need an infinitesimally thin surface $\delta S$ which spans the separation between the $\mathcal{L}(t)$ and $\mathcal{L}(t + \delta t)$:

Geometrically, the $S$ and the $\delta S$ can have rather complicated shapes in 3D, but topologically $S$ is a disk with outer boundary $\mathcal{L}(t)$ while $\delta S$ is an annulus with inner boundary $\mathcal{L}(t)$ and outer boundary $\mathcal{L}(t + \delta t)$, so that the combined surface $S + \delta S$ is a disk with outer boundary

![Diagram of a moving wire loop with a surface $S$ and an infinitesimally thin surface $\delta S$]
\[ \mathcal{L}(t + \delta t). \] Consequently, the magnetic flux through the loop changes with time from

\[ \Phi_B(t) = \iint_S \mathbf{B}(\mathbf{r}, t) \cdot d\mathbf{A} \]  \hspace{1cm} (14)

at time \( t \) to

\[ \Phi_B(t + \delta t) = \iint_{S + \delta S} \mathbf{B}(\mathbf{r}, t + \delta t) \cdot d\mathbf{A} = \iint_S \mathbf{B}(\mathbf{r}, t + \delta t) \cdot d\mathbf{A} + \iint_{\delta S} \mathbf{B}(\mathbf{r}, t + \delta t) \cdot d\mathbf{A} \]  \hspace{1cm} (15)

at time \( t + \delta t \). For infinitesimal \( \delta t \), the area of the ‘annulus’ \( \delta S \) is proportional to the \( \delta t \), so to first order in \( \delta t \), the magnetic flux changes by

\[ \delta \Phi_B = \Phi_B(t + \delta t) - \Phi_B(t) \]
\[ = \iint_S \left[ \mathbf{B}(\mathbf{r}; t + \delta t) - \mathbf{B}(\mathbf{r}; t) \right] \cdot d\mathbf{A} \]
\[ + \iint_{\delta S} \left[ \mathbf{B}(\mathbf{r}; t + \delta t) \approx \mathbf{B}(\mathbf{r}; t) \right] \cdot d\mathbf{A}, \] \hspace{1cm} (16)

hence the total time derivative of the flux decomposes into two terms:

\[ \frac{d}{dt} \Phi_B(t) = \iint_S \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \cdot d\mathbf{A} + \frac{1}{\delta t} \iint_{\delta S} \mathbf{B}(\mathbf{r}, t) \cdot d\mathbf{A}. \] \hspace{1cm} (17)

In terms of the Faraday’s Induction Law (1), this decomposition splits the net EMF into EMF due to changing magnetic field versus EMF due to the moving loop,

\[ \mathcal{E}_{\text{net}} = \mathcal{E}_{\partial \mathbf{B}/\partial t} + \mathcal{E}_{\text{motion}}. \] \hspace{1cm} (18)

\[ \mathcal{E}_{\partial \mathbf{B}/\partial t} = -\iint_S \frac{\partial \mathbf{B}(\mathbf{r}; t)}{\partial t} \cdot d^2\mathbf{A}, \] \hspace{1cm} (19)

\[ \mathcal{E}_{\text{motion}} = -\frac{1}{\delta t} \iint_{\delta S} \mathbf{B}(\mathbf{r}; t) \cdot d^2\mathbf{A}. \] \hspace{1cm} (20)
The EMF (19) due to time dependent magnetic field stems from the Maxwell’s Induction Equation (2). To see how this works, let’s use the Induction Equation to replace the time derivative of the magnetic field in eq. (19) with the curl of the induced electric field,

\[ \mathcal{E}_{\partial B/\partial t} = \int \int_S \left( -\frac{\partial B}{\partial t} = \nabla \times E_{\text{induced}} \right) \cdot d^2A, \quad (21) \]

and then use Stokes’s curl theorem for the area integral of a curl,

\[ \mathcal{E}_{\partial B/\partial t} = \int \int_S (\nabla \times E_{\text{induced}}) \cdot d^2A = \oint_{\mathcal{L}} E_{\text{induced}} \cdot d\vec{r}. \quad (22) \]

In other words, the EMF due to \( \partial B/\partial t \) stems from the non-potential electric work of the induced electric field, in accordance with the first term in eq. (8).

As to the motional EMF (20), it arises from the work of the Lorentz force on a moving loop, in accordance with the second term in eq. (8). To see how this works mathematically, we need to work out the infinitesimal area \( \delta S \) between the original and the moved loop is due to each piece of the wire moving with some velocity \( \mathbf{v}(r) \). Consider an infinitesimal piece \( d\vec{l} \) of the wire. Over infinitesimal time \( dt \), this piece is displaced by \( \delta r = \mathbf{v}\delta t \), and in the process it sweeps the area

\[ d^2A = \mathbf{v}\delta t \times d\vec{l}. \quad (23) \]
When the whole loop $\mathcal{L}$ moves with time, piece-by-piece, $d\vec{\ell}$ by $d\vec{\ell}$, it sweeps the net area

$$d\mathbf{A} = \oint_{\mathcal{L}} \mathbf{v}\delta t \times d\vec{\ell}. \quad (24)$$

In other words, the net area of the ‘annulus’ $\delta S$ is

$$\mathbf{A}[\delta S] = \oint_{\mathcal{L}} \mathbf{v}\delta t \times d\vec{\ell}. \quad (25)$$

Moreover, due to locality of this formula, we may use $d^2\mathbf{A} = \mathbf{v}\delta t \times d\vec{\ell}$ for area integrals over the $\delta S$ even when the integrand varies along the loop $\mathcal{L}$. In particular, on the RHS of eq. (20),

$$\iint_{\delta S} \mathbf{B}(\mathbf{r}, t) \cdot d^2\mathbf{A} = \oint_{\mathcal{L}} \mathbf{B} \cdot (\mathbf{v}\delta t \times d\vec{\ell}) \quad (26)$$

and hence

$$-\frac{1}{\delta t} \iint_{\delta S} \mathbf{B}(\mathbf{r}, t) \cdot d^2\mathbf{A} = -\oint_{\mathcal{L}} \mathbf{B} \cdot (\mathbf{v} \times d\vec{\ell}) = +\oint_{\mathcal{L}} d\vec{\ell} \cdot (\mathbf{v} \times \mathbf{B}) \quad (27)$$

where the second equality follows from the vector identity

for any 3 vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}),$$

in particular

$$-\mathbf{B} \cdot (\mathbf{v} \times d\vec{\ell}) = -d\vec{\ell} \cdot (\mathbf{B} \times \mathbf{v}) = +d\vec{\ell} \cdot (\mathbf{v} \times \mathbf{B}). \quad (28)$$
Altogether, the motional EMF becomes a loop integral

$$\mathcal{E}_{\text{motion}} = + \oint_{\mathcal{L}} d\vec{\ell} \cdot (\mathbf{v} \times \mathbf{B}), \quad (29)$$

precisely as in the second term in eq. (8).

Finally, let’s recombine the EMF (19) due to time-dependent magnetic field with the motional EMF (20). Combining eqs. (22) and (29), we arrive

$$\mathcal{E}_{\text{net}} = - \frac{d}{dt} \iint_{\mathcal{S}} \mathbf{B} \cdot d^2 \mathbf{A} = \oint_{\mathcal{L}} \left( \mathbf{E}_{\text{induced}} + \mathbf{v} \times \mathbf{B} \right) \cdot d\vec{\ell}, \quad (30)$$

precisely as promised in eq. (9).
**Torque on a Magnetic Moment**

Consider a closed loop $\mathcal{L}$ of wire carrying current $I$. In a magnetic field $\mathbf{B}$, every element $d\vec{\ell} = dr$ of the loop feels a force

$$d\mathbf{F} = I d\vec{\ell} \times \mathbf{B} \quad (31)$$

and therefore a torque (relative to the pivot point at the coordinate origin $\mathbf{r} = \vec{0}$)

$$d\vec{\tau} = \mathbf{r} \times d\mathbf{F} = \mathbf{r} \times (Idr \times \mathbf{B}). \quad (32)$$

In a uniform magnetic field $\mathbf{B}(x, y, z) \equiv \text{const}$ the net force on the closed wire loop is zero. Indeed,

$$\mathbf{F}^{\text{net}} = \oint_{\mathcal{L}} d\mathbf{F} = \oint_{\mathcal{L}} I\vec{\ell} \times \mathbf{B} = I \left( \oint_{\mathcal{L}} d\vec{\ell} \right) \times \mathbf{B} = 0 \quad (33)$$

because

$$\oint_{\mathcal{L}} d\vec{\ell} = \oint_{\mathcal{L}} dr = \mathbf{r}[\text{end of } \mathcal{L}] - \mathbf{r}[\text{start of } \mathcal{L}] = 0. \quad (34)$$

By the general rule of torques, vanishing net force implies that the net torque on the loop is the same relative to any pivot point. Indeed, consider the net torques relative to the pivot points $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\tau_{\text{rel. (1)}}^{\text{net}} = \oint_{\mathcal{L}} (\mathbf{r} - \mathbf{r}_1) \times d\mathbf{F}(\mathbf{r}), \quad \tau_{\text{rel. (2)}}^{\text{net}} = \oint_{\mathcal{L}} (\mathbf{r} - \mathbf{r}_2) \times d\mathbf{F}(\mathbf{r}),$$

$$\tau_{\text{rel. (1)}}^{\text{net}} - \tau_{\text{rel. (2)}}^{\text{net}} = \oint_{\mathcal{L}} ((\mathbf{r} - \mathbf{r}_1) - (\mathbf{r} - \mathbf{r}_2) = \mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{F}(\mathbf{r}) \quad (35)$$

$$= \mathbf{r}_2 - \mathbf{r}_1 \times \oint_{\mathcal{L}} d\mathbf{F}(\mathbf{r}) = \mathbf{r}_2 - \mathbf{r}_1 \times \mathbf{F}^{\text{net}},$$

hence $\mathbf{F}^{\text{net}} = \vec{0} \implies \tau_{\text{rel. (1)}}^{\text{net}} = \tau_{\text{rel. (2)}}^{\text{net}} \downarrow$.
Thus, relative to any pivot point, the net torque on the current-carrying loop in a uniform magnetic field is

\[ \vec{\tau}_{\text{net}} = \oint_{\mathcal{L}} \vec{r} \times (I\,d\vec{r} \times \vec{B}). \]  

(36)

**Theorem:** this torque obtains from the magnetic moment of the loop \( \vec{M} = I\text{Area} \) as

\[ \vec{\tau}_{\text{net}} = \vec{M} \times \vec{B}. \]  

(37)

**Note:** In vector form, the area of a closed loop obtains as a line integral

\[ \mathbf{A} [\mathcal{L}] = \frac{1}{2} \oint_{\mathcal{L}} \vec{r} \times d\vec{r}. \]  

(38)

For example, for a circular loop of radius \( R \),

\[ \vec{r} = (R \cos \phi, R \sin \phi, 0), \quad d\vec{r} = (-R \sin \phi, R \cos \phi, 0) \, d\phi, \]

\[ \vec{r} \times d\vec{r} = (0, 0, R^2) \, d\phi = R^2 \vec{i}_z \times d\phi, \]

\[ \text{hence } \mathbf{A} = \frac{R^2}{2} \vec{i}_z \times \oint d\phi = \frac{R^2}{2} \vec{i}_z \times 2\pi = \pi R^2 \vec{i}_z. \]  

(39)

Consequently, the magnetic moment vector of a current-carrying loop is

\[ \vec{M} = \frac{I}{2} \oint_{\mathcal{L}} \vec{r} \times d\vec{r}. \]  

(40)

**Proof:** In a uniform magnetic field \( \vec{B} = \text{const} \),

\[ d(\vec{r} \times (\vec{r} \times \vec{B})) = d\vec{r} \times (\vec{r} \times \vec{B}) + \vec{r} \times (d\vec{r} \times \vec{B}), \]  

(41)

hence integrating this complete differential over a closed loop \( \mathcal{L} \) always yields zero,

\[ \oint_{\mathcal{L}} \left( d\vec{r} \times (\vec{r} \times \vec{B}) + \vec{r} \times (d\vec{r} \times \vec{B}) \right) = 0. \]  

(42)
Let’s multiply this formula by \(-\frac{1}{2}I\) and add this to the torque formula (36):

\[
\vec{\tau}^\text{net} = \vec{\tau}^\text{net} + \vec{0} \\
= \oint_L \vec{r} \times (I\vec{d}r \times \vec{B}) - \frac{I}{2} \oint_L (d\vec{r} \times (\vec{r} \times \vec{B}) + \vec{r} \times (d\vec{r} \times \vec{B})) \\
= \frac{I}{2} \oint_L (2\vec{r} \times (d\vec{r} \times \vec{B}) - d\vec{r} \times (\vec{r} \times \vec{B}) - \vec{r}(d\vec{r} \times \vec{B})) \\
= \frac{I}{2} \oint_L (\vec{r} \times (d\vec{r} \times \vec{B}) - d\vec{r} \times (\vec{r} \times \vec{B})).
\]

(43)

Now let’s apply the antisymmetry \(\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}\) and the Jacobi identity

\[
\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}
\]

(44)
of the cross product to the integrand on the last line of eq. (43):

\[
\begin{align*}
\vec{r} \times (d\vec{r} \times \vec{B}) - d\vec{r} \times (\vec{r} \times \vec{B}) &= -\vec{r} \times (\vec{B} \times d\vec{r}) - d\vec{r} \times (\vec{r} \times \vec{B}) \\
&= -(\vec{r} \times (\vec{B} \times d\vec{r})) - d\vec{r} \times (\vec{r} \times \vec{B}) \\
&= +(\vec{r} \times d\vec{r}) \times \vec{B}.
\end{align*}
\]

(45)

Consequently, the net torque becomes

\[
\vec{\tau}^\text{net} = \frac{I}{2} \oint_L (\vec{r} \times d\vec{r}) \times \vec{B} = \frac{I}{2} \left( \oint_L \vec{r} \times d\vec{r} \right) \times \vec{B} = \mathbf{M} \times \vec{B}.
\]

(46)

Quod erat demonstrandum.