Dirac Matrices and Lorentz Spinors

**Background:** In 3D, the spinor \( j = \frac{1}{2} \) representation of the Spin(3) rotation group is constructed from the Pauli matrices \( \sigma^x, \sigma^y, \) and \( \sigma^z \), which obey both commutation and anticommutation relations

\[
[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k \quad \text{and} \quad \{\sigma^i, \sigma^j\} = 2\delta^{ij} \times 1_{2 \times 2}.
\]  

Consequently, the spin matrices

\[
S = -\frac{i}{2}\sigma \times \sigma = \frac{1}{2}\sigma
\]

commute with each other as angular momenta, \( [S^i, S^j] = i\epsilon^{ijk}S^k \), so they represent the generators of the rotation group. Moreover, under finite rotations \( R(\phi, \mathbf{n}) \) represented by

\[
M(R) = \exp(-i\phi \mathbf{n} \cdot \mathbf{S}),
\]

the spin matrices transform into each other as components of a 3–vector,

\[
M^{-1}(R)S^i M(R) = R^{ij} S^j.
\]

In this note, I shall generalize this construction to the Dirac spinor representation of the Lorentz symmetry Spin(3, 1).

**Dirac Matrices** are 4 anti-commuting \( 4 \times 4 \) matrices \( \gamma^\mu \),

\[
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \times 1_{4 \times 4}.
\]

The specific form of these matrices is not important — as long as they obey the anticommutation relations (5) — and different books use different conventions. In my class I shall follow the same convention as the Peskin & Schroeder textbook, namely the Weyl convention where in \( 2 \times 2 \) block notations

\[
\gamma^0 = \begin{pmatrix} 0 & 1_{2 \times 2} \\ 1_{2 \times 2} & 0 \end{pmatrix}, \quad \tilde{\gamma} = \begin{pmatrix} 0 & +\sigma \\ -\sigma & 0 \end{pmatrix}.
\]

Note that the \( \gamma^0 \) matrix is hermitian while the \( \gamma^1, \gamma^2, \) and \( \gamma^3 \) matrices are anti-hermitian.
Lorentz spin matrices.
Given the Dirac matrices obeying the anticommutation relations (5), we may define the spin matrices as

\[ S_{\mu\nu} = -S_{\nu\mu} \overset{\text{def}}{=} i[\gamma^\mu, \gamma^\nu]. \quad (7) \]

These matrices obey the same commutation relations as the generators \( \hat{J}^{\mu\nu} = -\hat{J}^{\nu\mu} \) of the continuous Lorentz group. Moreover, their commutation relations with the Dirac matrices \( \gamma^\mu \) are similar to the commutation relations of the \( \hat{J}^{\mu\nu} \) with a Lorentz vector such as \( \hat{P}^\mu \).

Lemma:

\[ [\gamma^\lambda, S^{\mu\nu}] = ig^{\lambda\mu} \gamma^\nu - ig^{\lambda\nu} \gamma^\mu. \quad (8) \]

Proof: Combining the definition (7) of the spin matrices as commutators with the anticommutation relations (5), we have

\[ \gamma^\mu \gamma^\nu = \frac{1}{2} \{\gamma^\mu, \gamma^\nu\} + \frac{1}{2} [\gamma^\mu, \gamma^\nu] = g^{\mu\nu} \mathbf{1}_{4\times4} - 2iS^{\mu\nu}. \quad (9) \]

Since the unit matrix commutes with everything, we have

\[ [X, S^{\mu\nu}] = \frac{i}{2} [X, \gamma^\mu \gamma^\nu] \quad \text{for any matrix } X, \quad (10) \]

and the commutator on the RHS may often be obtained from the Leibniz rules for the commutators or anticommutators:


In particular,

\[ [\gamma^\lambda, \gamma^\mu \gamma^\nu] = \{\gamma^\lambda, \gamma^\mu\} \gamma^\nu - \gamma^\mu \{\gamma^\lambda, \gamma^\nu\} = 2g^{\lambda\mu} \gamma^\nu - 2g^{\lambda\nu} \gamma^\mu \quad (12) \]

and hence

\[ [\gamma^\lambda, S^{\mu\nu}] = \frac{i}{2} [\gamma^\lambda, \gamma^\mu \gamma^\nu] = ig^{\lambda\mu} \gamma^\nu - ig^{\lambda\nu} \gamma^\mu. \quad (13) \]

Quod erat demonstrandum.
**Theorem:** The \( S^{\mu\nu} \) matrices commute with each other like Lorentz generators,

\[
[S^{\kappa\lambda}, S^{\mu\nu}] = i g^{\lambda\mu} S^{\kappa\nu} - i g^{\kappa\nu} S^{\mu\lambda} - i g^{\lambda\nu} S^{\kappa\mu} + i g^{\kappa\mu} S^{\lambda\nu}. \tag{14}
\]

**Proof:** Again, we use the Leibniz rule and eq. (9):

\[
\left[ \gamma^\kappa \gamma^\lambda, S^{\mu\nu} \right] = \gamma^\kappa \left[ \gamma^\lambda, S^{\mu\nu} \right] + \left[ \gamma^\kappa, S^{\mu\nu} \right] \gamma^\lambda \\
= \gamma^\kappa \left( i g^{\lambda\mu} \gamma^\nu - i g^{\lambda\nu} \gamma^\mu \right) + (i g^{\kappa\nu} \gamma^\mu - i g^{\kappa\mu} \gamma^\nu) \gamma^\lambda \\
= i g^{\lambda\mu} \gamma^\kappa \gamma^\nu - i g^{\kappa\nu} \gamma^\mu \gamma^\lambda - i g^{\lambda\nu} \gamma^\kappa \gamma^\mu + i g^{\kappa\mu} \gamma^\nu \gamma^\lambda \\
= i g^{\lambda\mu} (g^{\kappa\nu} - 2i S^{\kappa\nu}) - i g^{\kappa\nu} (g^{\lambda\mu} + 2i S^{\lambda\mu}) \\
- i g^{\lambda\nu} (g^{\kappa\mu} - 2i S^{\kappa\mu}) + i g^{\kappa\mu} (g^{\lambda\nu} + 2i S^{\lambda\nu}) \\
= 2g^{\lambda\mu} S^{\kappa\nu} - 2g^{\kappa\nu} S^{\lambda\mu} - 2g^{\lambda\nu} S^{\kappa\mu} + 2g^{\kappa\mu} S^{\lambda\nu}, \tag{15}
\]

and hence

\[
[S^{\kappa\lambda}, S^{\mu\nu}] = \frac{1}{2} \left[ \gamma^\kappa \gamma^\lambda, S^{\mu\nu} \right] = i g^{\lambda\mu} S^{\kappa\nu} - i g^{\kappa\nu} S^{\mu\lambda} - i g^{\lambda\nu} S^{\kappa\mu} + i g^{\kappa\mu} S^{\lambda\nu}. \tag{16}
\]

*Quod erat demonstrandum.*

In light of this theorem, the \( S^{\mu\nu} \) matrices represent the Lorentz generators \( \hat{J}^{\mu\nu} \) in a 4-component spinor multiplet.

**Finite Lorentz transforms:**

Any continuous Lorentz transform — a rotation, or a boost, or a product of a boost and a rotation — obtains from exponentiating an infinitesimal symmetry

\[
X^{\mu} = X^\mu + \epsilon^{\mu\nu} X_\nu \tag{17}
\]

where the infinitesimal \( \epsilon^{\mu\nu} \) matrix is antisymmetric when both indices are raised (or both lowered), \( \epsilon^{\mu\nu} = -\epsilon^{\nu\mu} \). Thus, the \( L_\nu^\mu \) matrix of any continuous Lorentz transform is a matrix exponential

\[
L_\nu^\mu = \exp(\Theta)_\nu^\mu \equiv \delta_\nu^\mu + \Theta^\mu_\nu + \frac{1}{2} \Theta^\mu_\lambda \Theta^\lambda_\nu + \frac{1}{6} \Theta^\mu_\lambda \Theta^\lambda_\kappa \Theta^\kappa_\nu + \cdots \tag{18}
\]

of some matrix \( \Theta \) that becomes antisymmetric when both of its indices are raised or lowered, \( \Theta^{\mu\nu} = -\Theta^{\nu\mu} \). Note however that in the matrix exponential (18), the first index of \( \Theta \) is raised.
while the second index is lowered, so the antisymmetry condition becomes \((g\Theta)\top = -(g\Theta)\) instead of \(\Theta\top = -\Theta\).

The Dirac spinor representation of the finite Lorentz transform (18) is the 4 × 4 matrix

\[
M_D(L) = \exp\left(-\frac{i}{2} \Theta_{\alpha\beta} S^\alpha{}^\beta\right).
\] (19)

The group law for such matrices

\[
\forall L_1, L_2 \in SO^+(3, 1), \quad M_D(L_2 L_1) = M_D(L_2) M_D(L_1)
\] (20)

follows automatically from the \(S^{\mu\nu}\) satisfying the commutation relations (14) of the Lorentz generators, so I am not going to prove it. Instead, let me show that when the Dirac matrices \(\gamma^\mu\) are sandwiched between the \(M_D(L)\) and its inverse, they transform into each other as components of a Lorentz 4–vector,

\[
M_D^{-1}(L) \gamma^\mu M_D(L) = L^\mu{}^\nu \gamma^\nu.
\] (21)

This formula makes the Dirac equation transform covariantly under the Lorentz transforms.

Proof: In light of the exponential form (19) of the matrix \(M_D(L)\) representing a finite Lorentz transform in the Dirac spinor multiplet, let’s use the multiple commutator formula (AKA the Hadamard Lemma): for any 2 matrices \(F\) and \(H\),

\[
\exp(-F)H \exp(+F) = H + [H, F] + \frac{1}{2} [[[H, F], F], F] + \cdots.
\] (22)

In particular, let \(H = \gamma^\mu\) while \(F = -\frac{i}{2} \Theta_{\alpha\beta} S^\alpha{}^\beta\) so that \(M_D(L) = \exp(+F)\) and \(M_D^{-1}(L) = \exp(-F)\). Consequently,

\[
M_D^{-1}(L) \gamma^\mu M_D(L) = \gamma^\mu + [\gamma^\mu, F] + \frac{1}{2} [[[\gamma^\mu, F], F], F] + \cdots
\] (23)

where all the multiple commutators turn out to be linear combinations of the Dirac matrices.
Indeed, the single commutator here is
\[
[\gamma^\mu, F] = -\frac{i}{2} \Theta_{\alpha\beta} [\gamma^\mu, S_{\alpha\beta}] = \frac{1}{2} \Theta_{\alpha\beta} (g^{\mu \alpha} \gamma^\beta - g^{\mu \beta} \gamma^\alpha) = \Theta_{\alpha\beta} g^{\mu \alpha} \gamma^\beta = \Theta^\mu_\lambda \gamma^\lambda,
\] (24)
while the multiple commutators follow by iterating this formula:
\[
[[[\gamma^\mu, F], F], F] = \Theta^\mu_\lambda \Theta^\lambda_\rho \Theta^\rho_\gamma \gamma^\nu, \quad \ldots
\] (25)
Combining all these commutators as in eq. (23), we obtain
\[
M^{-1}_D \gamma^\mu M_D = \gamma^\mu + [\gamma^\mu, F] + \frac{1}{2} [[[\gamma^\mu, F], F], F] + \ldots
\]
\[
= \gamma^\mu + \Theta^\mu_\nu \gamma^\nu + \frac{1}{2} \Theta^\mu_\nu \Theta^\nu_\gamma \gamma^\nu + \frac{1}{6} \Theta^\mu_\nu \Theta^\nu_\rho \Theta^\rho_\gamma \gamma^\nu + \ldots
\]
\[
= \left( \delta^\mu_\nu + \Theta^\mu_\nu + \frac{1}{2} \Theta^\mu_\nu \Theta^\nu_\gamma + \frac{1}{6} \Theta^\mu_\nu \Theta^\nu_\rho \Theta^\rho_\gamma + \ldots \right) \gamma^\nu
\]
\[
\equiv L^\mu_\nu \gamma^\nu.
\] (26)
\textit{Quod erat demonstrandum.}

**Dirac Equation**

The Dirac spinor field \( \Psi(x) \) has 4 complex components \( \Psi_\alpha(x) \) arranged in a column vector
\[
\Psi(x) = \begin{pmatrix}
\Psi_1(x) \\
\Psi_2(x) \\
\Psi_3(x) \\
\Psi_4(x)
\end{pmatrix}.
\] (27)
Under continuous Lorentz symmetries \( x'\mu = L^\mu_\nu x^\nu \), the spinor field transforms as
\[
\Psi'(x') = M_D(L) \Psi(x).
\] (28)
The classical field equation for the free spinor field is the Dirac equation — a first-order differential equation
\[
(i \gamma^\mu \partial_\mu - m) \Psi(x) = 0.
\] (29)
The Dirac equation implies the Klein–Gordon equation for each component \( \Psi_\alpha(x) \). Indeed,
if $\Psi(x)$ obey the Dirac equation, then

$$(-i\gamma^\nu \partial_\nu - m)(i\gamma^\mu \partial_\mu - m)\Psi(x) = 0$$

(30)

where the differential operator on the LHS is the Klein–Gordon $m^2 + \partial^2$ times a unit matrix. Indeed,

$$(-i\gamma^\nu \partial_\nu - m)(i\gamma^\mu \partial_\mu - m) = m^2 + \gamma^\nu \gamma^\mu \partial_\nu \partial_\mu = m^2 + \frac{1}{2} \{\gamma^\mu, \gamma^\nu\} \partial_\nu \partial_\mu = m^2 + g^{\mu\nu} \partial_\mu \partial_\nu.$$  

(31)

The Dirac equation transforms covariantly under the Lorentz symmetries — its LHS transforms exactly like the spinor field itself.

**Proof:** Note that since the Lorentz symmetries involve the $x^\mu$ coordinates as well as the spinor field components, the LHS of the Dirac equation becomes

$$(i\gamma^\mu \partial'_\mu - m)\Psi'(x')$$

(32)

where

$$\partial'_\mu \equiv \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \times \frac{\partial}{\partial x^\nu} = (L^{-1})^\nu_\mu \times \partial_\nu.$$  

(33)

Consequently,

$$\partial'_\mu \Psi'(x') = (L^{-1})^\nu_{\mu} M_D(L) \partial_\nu \Psi(x)$$

(34)

and hence

$$\gamma^\mu \partial'_\mu \Psi'(x') = (L^{-1})^\nu_{\mu} \gamma^\mu M_D(L) \partial_\nu \Psi(x).$$

(35)

But according to eq. (23),

$$M_D^{-1}(L) \gamma^\mu M_D(L) = L^\mu_\nu \gamma^\nu \implies \gamma^\mu M_D(L) = L^\mu_\nu \times M_D(L) \gamma^\nu \implies (L^{-1})^\nu_{\mu} \gamma^\mu M_D(L) = M_D(L) \gamma^\nu,$$

so

$$\gamma^\mu \partial'_\mu \Psi'(x') = M_D(L) \times \gamma^\nu \partial_\nu \Psi(x).$$

(36)

(37)

 Altogether,

$$\begin{align*}
(i\gamma^\mu \partial_\mu - m)\Psi(x) & \xrightarrow{\text{Lorentz}} (i\gamma^\mu \partial'_\mu - m)\Psi'(x') = M_D(L) \times (i\gamma^\mu \partial_\mu - m)\Psi(x),
\end{align*}$$

(38)

which proves the covariance of the Dirac equation. *Quod erat demonstrandum.*