Glashow–Weinberg–Salam Theory

Glashow–Weinberg–Salam theory describing both weak and electromagnetic interactions is the $SU(2)_W \times U(1)_Y$ gauge theory spontaneously broken down to the $U(1)_{EM}$. Out of 4 gauge fields $W^a_\mu$ ($a = 1, 2, 3$) and $B_\mu$, one linear combination remains massless and gives rise to the electromagnetism, while 3 other linear combinations become massive and give rise to the weak interaction.

The key to the spontaneous breakdown of the electroweak gauge symmetry is the doublet of complex fields $H_\alpha$ ($\alpha = 1, 2$) called the Higgs fields. The $SU(2)_W \times U(1)_Y$ quantum numbers of these fields are $(2, +\frac{1}{2})$; that is, they form a doublet of the $SU(2)_W$ and have the $U(1)_Y$ hypercharge $y = +\frac{1}{2}$. Thus,

\[
D_\mu H_\alpha(x) = \partial_\mu H_\alpha + \frac{ig_2}{2} \tau^a_{\alpha\beta} W^a_\mu(x) H_\beta(x) + \frac{ig_1}{2} B_\mu H_\alpha
\]

where $g_2$ is the $SU(2)_W$ gauge coupling and $g_1$ is the $U(1)_Y$ gauge coupling.

The gauge fields $W^a_\mu$ and $B_\mu$ and the Higgs fields $H_\alpha$ are the only bosonic fields of the GSW theory. There are also 90 fermionic fields describing the quarks and the leptons, but we shall study them later this semester. The net Lagrangian of the theory is

\[
\mathcal{L} = -\frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + D_\mu H^\dagger D^\mu H - \frac{\lambda}{2} \left( H^\dagger H - \frac{v^2}{2} \right)^2 + \text{fermionic terms (2)}
\]

where

\[
B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu,
W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g_2 \epsilon^{abc} W^b_\mu W^c_\nu,
H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}, \quad H^\dagger = (H_1^*, H_2^*),
\]

and $D_\mu H$, $D_\mu H^\dagger$ are similar row/column vector forms of $D_\mu H_\alpha$ and $D_\mu H^*_\alpha$, cf. eq. (1). The scalar potential $V = \frac{\lambda}{2} \left( H^\dagger H - \frac{v^2}{2} \right)^2$ has a local maximum rather than a minimum at $H = 0$, while its minima form a spherical shell $H^\dagger H = \frac{v^2}{2}$ in the scalar field space $\mathbb{C}^2 = \mathbb{R}^4$. All such minima are related to each other by the gauge symmetry, so without loss of generality we
assume the Higgs fields have Vacuum Expectation Values (VEVs)

\[ \langle H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} . \]  

(4)

Note that this expectation value breaks 3 out of 4 gauge symmetries of the theory, but one combination of the $U(1)_Y$ and an $U(1)$ subgroup of the $SU(2)_2$ remains unbroken. Indeed, the $U(1)_Y$ symmetry $\exp(i\Theta(x)\hat{Y})$ acts on the Higgs fields as $H(x) \rightarrow \exp(iy\Theta(x))H(x) = \exp(\frac{i}{2}\Theta(x))H(x)$ since $H$ has $y = +\frac{1}{2}$, while the $SU(2)$ symmetry $\exp(i\Theta(x)\hat{T}^3)$ — for the same $\Theta(x)$ — acts on the $SU(2)$ double $H$ as $H(x) \rightarrow \exp(\frac{i}{2}\Theta(x)\tau^3)H(x)$. Combining the two symmetries, we have

\[ H(x) \rightarrow \exp(\frac{i}{2}\Theta(x))\exp(\frac{i}{2}\Theta(x)\tau^3)H(x) = \begin{pmatrix} e^{i\Theta(x)} & 0 \\ 0 & 1 \end{pmatrix} H(x), \]  

which indeed leaves the vacuum expectation value (4) invariant. Thus, the $U(1)$ subgroup of the electroweak $SU(2)_W \times U(1)_Y$ generated by the operator

\[ \hat{Q} = \hat{Y} + \hat{T}^3 \]  

remains unbroken. Physically, this subgroup is the $U(1)_Q$ gauge symmetry of the electromagnetism and $\hat{Q}$ is the electric charge operator (or rather electric charge in units of $e$).

We shall see in a moment that one linear combination of the four $SU(2)_W \times U(1)_Y$ gauge fields corresponding to the $\hat{Q}$ generator remains massless while the other 3 combinations become massive via the Higgs mechanism. The same mechanism also eliminates 3 scalar fields, which becomes the longitudinal components of the 3 massive vector fields. Since the 2 complex Higgs fields are equivalent to 4 real scalars, we end up with $4 - 3 = 1$ physical scalar field $\hat{h}(x)$; its quanta — called the physical Higgs particles — were experimentally discovered at the LHC this year.

The simplest way to see how this works is to fix the unitary gauge for the spontaneously broken symmetries. Note that any complex doublet $H(x)$ can be $SU(2)$–rotated to

\[ H'(x) = U(x)H(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \tilde{h}(x) \end{pmatrix} \]  

for a real $\tilde{h}(x) \geq 0$. This gauge transform would be singular for $H(x) \approx 0$ but it is nice and smooth for $H(x)$ in the vicinity of the vacuum expectation value (4), so we may use it to fix
In terms of this shifted field,

\[ H^\dagger H - \frac{v^2}{2} = \frac{(v + h)^2}{2} - \frac{v^2}{2} = vh + \frac{1}{2}h^2, \]  

so the scalar potential becomes

\[ V(h) = \frac{\lambda}{2} \left( H^\dagger H - \frac{v^2}{2} \right)^2 = \frac{\lambda}{2}(vh + \frac{1}{2}h^2)^2 = \frac{\lambda v^2}{2} \times h^2 + \frac{\lambda v}{2} \times h^3 + \frac{\lambda}{8} \times h^4 \]  

with a positive mass^2 = \lambda v^2 > 0 for the physical Higgs field.

The mass terms for the vector fields emerge from the kinetic term \( D_\mu H^\dagger D^\mu H \) for the Higgs doublets. Indeed, in the unitary gauge

\[ D_\mu H = \frac{1}{\sqrt{2}} \left( \frac{i}{2} g_2 (W^1_\mu - iW^2_\mu) \tilde{h} \right) = \frac{1}{\sqrt{2}} \left( \frac{i}{2} g_2 (W^1_\mu - iW^2_\mu) (v + h) \right) \]  

and hence

\[ D_\mu H^\dagger D^\mu H = \frac{1}{2} \left| \partial_\mu \tilde{h} + \frac{i}{2} (g_1 B_\mu - g_2 W^3_\mu) (v + h) \right|^2 \]  

\[ = \frac{1}{2} (\partial_\mu h)^2 + \frac{(v + h)^2}{8} \left( g_1 B_\mu - g_2 W^3_\mu \right)^2 + \frac{g_2^2 (v + h)^2}{8} \left( (W^1_\mu)^2 + (W^2_\mu)^2 \right). \]  

The first term on the last line here is the kinetic term for the physical Higgs field while the rest are the mass terms for the vector fields and also their interactions with the physical Higgs field \( h(x) \). In particular, the vector mass terms obtain from truncating \((v + h(x))^2\) coefficients.
to simply $v^2$, thus

$$L_{masses}^{\text{vector}} = \frac{g_2^2 v^2}{8} \times \left( (W_{\mu}^1)^2 + (W_{\mu}^2)^2 \right) + \frac{v^2}{8} \times (g_1 B_{\mu} - g_2 W_{\mu}^3)^2. \quad (13)$$

In particular, the $W_{\mu}^1$ and $W_{\mu}^2$ vector fields have masses

$$M_{W}^2 = \frac{g_2^2 v^2}{4} \implies M_{W} = \frac{g_2 v}{2} \quad (14)$$

while the $W_{\mu}^3$ and $B_{\mu}$ vector fields have a $2 \times 2$ mass matrix

$$M^2 = \frac{v^2}{4} \begin{pmatrix} g_2^2 & -g_2 g_1 \\ -g_2 g_1 & g_1^2 \end{pmatrix}. \quad (15)$$

This matrix has eigenvalues

$$M_{Z}^2 = \frac{(g_2^2 + g_1^2) v^2}{4} \quad \text{and} \quad M_{A}^2 = 0 \quad (16)$$

— as promised, there is one massless vector field — while the mass eigenstates correspond to vector fields

$$Z_{\mu}(x) = \cos \theta \times W_{\mu}^3(x) - \sin \theta \times B_{\mu}(x),$$

$$A_{\mu}(x) = \sin \theta \times W_{\mu}^3(x) + \cos \theta \times B_{\mu}(x), \quad (17)$$

where

$$\theta = \arctan \frac{g_1}{g_2} \quad (18)$$

is the weak mixing angle or the Weinberg angle; experimentally, $\sin^2 \theta \approx 0.23$.

Physically, the $A_{\mu}(x)$ is the EM field whose quanta are massless photons, the $Z_{\mu}(x)$ is the neutral weak field whose quanta are $Z^0$ particles of mass $M_Z \approx 91$ GeV, and the $W_{\mu}^{1,2}(x)$ — or rather their linear combinations

$$W_{\mu}^+(x) = \frac{W_{\mu}^1(x) + iW_{\mu}^2(x)}{\sqrt{2}} \quad \text{and} \quad W_{\mu}^-(x) = \frac{W_{\mu}^1(x) - iW_{\mu}^2(x)}{\sqrt{2}} \quad (19)$$

— are the charged weak fields (electric charges $q = \pm 1$) whose quanta are the $W^+$ and $W^-$ particles of mass $M_W \approx 80$ GeV. The experimentally found mass ratio between the $W^\pm$ and
$Z^0$ particles gives us the value of the weak mixing angle:

$$
\frac{M_W^2}{M_Z^2} = \frac{g_2^2}{g_2^2 + g_1^2} = \frac{1}{1 + \tan^2 \theta} = \cos^2 \theta \implies \cos^2 \theta \approx 0.77 \implies \sin^2 \theta \approx 0.23. \quad (20)
$$

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Now let’s find the currents to which the vector fields $W^\pm_\mu$, $Z_\mu$, and $A_\mu$ couple and the strengths of those couplings. Of particular importance is the EM coupling strength $e$ since it acts as the unit of the conventionally normalized electric charge, so we would like to relate it to the original $SU(2)_W \times U(1)_Y$ couplings $g_2$ and $g_1$. But the weak currents and couplings are also important.

Our starting point is the $SU(2)_W \times U(1)_Y$ symmetry currents $J^Y_\mu$, $J^{T1}_\mu$, $J^{T2}_\mu$, $J^{T3}_\mu$ of the fermionic fields. Without going into the details of these currents — we shall do that later this semester — we can say that the original gauge fields $B_\mu(x)$ and $W^a_\mu(x)$ couple to these currents according to

$$
\mathcal{L}_{\text{net}} \supset \mathcal{L}_{\text{current}} = -g_2 W^1_\mu \times J^T_1 - g_2 W^2_\mu \times J^T_2 - g_2 W^3_\mu \times J^T_3 - g_1 B_\mu \times J^Y_\mu. \quad (21)
$$

Now let’s relate the original gauge fields to the vector fields of definite masses and electric charges. Inverting eqs. (19) and (17), we obtain

\begin{align*}
W^1_\mu &= \frac{1}{\sqrt{2}} \times W^-_\mu + \frac{1}{\sqrt{2}} \times W^+_\mu, \\
W^2_\mu &= \frac{i}{\sqrt{2}} \times W^-_\mu - \frac{i}{\sqrt{2}} \times W^+_\mu, \\
W^3_\mu &= \cos \theta \times Z_\mu + \sin \theta \times A_\mu, \\
B_\mu &= -\sin \theta \times Z_\mu + \cos \theta \times A_\mu.
\end{align*}

Plugging these formulae into eq. (21), we find

\begin{align*}
\mathcal{L}_{\text{current}} &= -\frac{g_2}{\sqrt{2}} W^-_\mu \times (J^T_1 - i J^T_2) - \frac{g_2}{\sqrt{2}} W^+_\mu \times (J^T_1 + i J^T_2) \\
&\quad - Z_\mu \times (g_2 \cos \theta J^T_3 - g_1 \sin \theta J^Y_\mu) - A_\mu \times (g_2 \sin \theta J^T_3 + g_1 \cos \theta J^Y_\mu) \\
&\equiv -\frac{g_2}{\sqrt{2}} (W^+_\mu \times J^-_\mu + W^-_\mu \times J^+\mu) - \tilde{g} Z_\mu \times J^Y_\mu - e A_\mu \times J^Y_\mu \quad (23)
\end{align*}
where

\[ J^+ = j_{T1}^\mu - ij_{T2}^\mu, \quad J^- = j_{T1}^\mu + ij_{T2}^\mu; \tag{24} \]

are the charged weak currents,

\[ \tilde{\gamma} \times j_Z^\mu = g_2 \cos \theta j_{T3}^\mu - g_1 \sin \theta J_Y^\mu \tag{25} \]

is the neutral weak current (times the neutral weak coupling constant), and

\[ e \times J_{\text{EM}}^\mu = g_2 \sin \theta j_{T3}^\mu + g_1 \cos \theta J_Y^\mu \tag{26} \]

is the (conventionally normalized) electric current. Note that on the right hand side of this formula \( g_1 \cos \theta = g_2 \sin \theta \) because of the way the weak mixing angle \( \theta \) is related to the gauge couplings, \( \tan \theta = g_1/g_2 \), cf. eq. (18). Consequently, we may identify

\[ e = g_2 \sin \theta = g_1 \cos \theta \implies \frac{1}{e^2} = \frac{1}{g_2^2} \left( \frac{1}{\sin^2 \theta} = 1 + \frac{1}{\tan^2 \theta} \right) = \frac{1}{g_2^2} + \frac{1}{g_1^2} \tag{27} \]

and

\[ J_{\text{EM}}^\mu = j_{T3}^\mu + J_Y^\mu. \tag{28} \]

Note that this current does not depend on the gauge couplings or \( \theta \); instead, it’s the current of the electric charge operator \( \hat{Q} = \hat{T}^3 + \hat{Y} \) which is the generator of the unbroken \( U(1)_{\text{EM}} \) gauge symmetry. Naturally, the EM field \( A_\mu(x) \) — which is the gauge field of that \( U(1)_{\text{EM}} \) — should couple to precisely this symmetry current.

On the other hand, the \( Z_\mu \) is the gauge field of a spontaneously broken symmetry, so the specific combination of the symmetry currents that couples to the \( Z_\mu \) depends on the weak mixing angle. Indeed, the coefficients of the two terms on the RHS of eq. (25) are quite different and their ratio depends on \( g_1/g_2 \); specifically,

\[ g_2 \cos \theta = \frac{g_2^2}{\sqrt{g_2^2 + g_1^2}} = \sqrt{g_2^2 + g_1^2} \times \cos^2 \theta, \]

\[ g_1 \cos \theta = \frac{g_1^2}{\sqrt{g_2^2 + g_1^2}} = \sqrt{g_2^2 + g_1^2} \times \sin^2 \theta, \tag{29} \]

\[ \frac{g_1 \sin \theta}{g_2 \cos \theta} = \tan^2 \theta. \]
Consequently, we may identify
\[
\tilde{g} = \sqrt{g_2^2 + g_1^2} = \frac{g_2}{\cos \theta} = \frac{g_1}{\sin \theta} = \frac{e}{\sin \theta \cos \theta}
\] (30)
and then the neutral weak current becomes
\[
J_{\mu}^Z = \cos^2 \theta \times J_{T 3}^\mu - \sin^2 \theta \times J_Y^\mu = J_{T 3}^\mu - \sin^2 \theta \times (J_{T 3}^\mu + J_Y^\mu)
= J_{T 3}^\mu - \sin^2 \theta \times J_{EM}^\mu.
\] (31)

Note that the weak couplings \(g_2\) and \(\tilde{g}\) are larger than the EM coupling \(e\). Consequently, at high energies much larger than the masses of \(W\) and \(Z\) particles, the weak interactions are not weak at all — they are stronger than the EM interactions. But at low energies, the \(\beta\)-decays and other processes mediated by the virtual \(W^\pm\) or \(Z^0\) are suppressed by the very small factors \(E^2/M_W^2\) of \(E^2/M_Z^2\) and it is those factors that make the weak interactions weak. I’ll explain how this works in November, when we study the Feynman diagrams. For the moment, let me simply outline the effective Fermi theory for the low-energy weak interactions.

The low-energy processes do not produce any real \(W^\pm\) or \(Z^0\) particles, but they do involve the weak currents \(J^{\pm \mu}\) and \(J_X^\mu\), which in turn give rise to the small \(W^{\pm}(x)\) and \(Z_\mu(x)\) fields. To see how this works at the classical level, consider the electroweak Lagrangian
\[
\mathcal{L} = -\frac{1}{2} W^-_{\mu \nu} W^{+ \mu \nu} + M_W^2 W^-_{\mu} W^{+ \mu} - \frac{g_2}{\sqrt{2}} (W^+_{\mu} \times J^-_{\mu} + W^-_{\mu} \times J^{+ \mu})
- \frac{1}{4} Z_{\mu \nu} Z^{\mu \nu} + \frac{M_Z^2}{2} Z_{\mu} Z^{\mu} - \tilde{g} J_Z^\mu Z_{\mu}
\] (32)

To the leading order of perturbation theory we may neglect the non-abelian terms here as well as interactions of the vector fields with the physical Higgs field \(h\). Moreover, at low energy-momenta \(k^\mu \ll M_W, M_Z\), the kinetic terms for the \(W^{\pm}_{\mu}\) and \(Z_{\mu}\) fields are much smaller than the respective mass terms. Consequently, we may approximate
\[
\mathcal{L} \approx M_W^2 W^-_{\mu} W^{+ \mu} - \frac{g_2}{\sqrt{2}} (W^+_{\mu} \times J^-_{\mu} + W^-_{\mu} \times J^{+ \mu}) + \frac{M_Z^2}{2} Z_{\mu} Z^{\mu} - \tilde{g} J_Z^\mu Z_{\mu}.
\] (33)
The field equations stemming from this approximate Lagrangian are simply

\[ M_W^2 \times W^{\pm \mu} \approx \frac{g_2}{\sqrt{2}} \times J^{\pm \mu} \quad \text{and} \quad M_Z^2 \times Z^\mu \approx \tilde{g} \times J^\mu_Z. \]  

(34)

Solving these equations and plugging the solutions back into the Lagrangian (33), we obtain the effective current-current Lagrangian for the low-energy weak interactions,

\[ \mathcal{L}_{\text{effective}}^{\text{weak}} = -\frac{g_2^2}{2M_W^2} \times J^-_{\mu} J^{+\mu} - \frac{\tilde{g}^2}{2M_Z^2} \times J_{Z\mu} J^\mu_Z. \]  

(35)

This effective Lagrangian is called the Fermi Lagrangian — and the corresponding effective theory of weak interactions is called the Fermi Theory — since Enrico Fermi wrote it down back in 1933. Or rather, he wrote down

\[ \mathcal{L} = -2\sqrt{2}G \times J^-_{\mu} J^{+\mu} \]  

(36)

since only the charged-current weak interactions we known in those days, and the weak coupling \( G \) was an input parameter to be determined experimentally. Today \( G \) is called the Fermi constant and we know how to relate it to the vacuum expectation value of the Higgs field:

\[ G = \frac{1}{4\sqrt{2}} \frac{g_2^2}{M_W^2} = \frac{1}{\sqrt{2}} \times \frac{1}{v^2}. \]  

(37)

The neutral-coupling weak interactions have a separate Fermi-like constant, but in the GWS electroweak theory it has exactly the same value as the Fermi constant for the charged-current weak interactions:

\[ \mathcal{L}_{\text{effective}}^{\text{weak}} = -2\sqrt{2}G \left( J^-_{\mu} J^{+\mu} + \rho \times J_{Z\mu} J^\mu_Z \right) \quad \text{for} \ \rho = 1. \]  

(38)

Indeed,

\[ G \times \rho = \frac{1}{4\sqrt{2}} \frac{\tilde{g}^2}{M_Z^2} \quad \implies \quad \rho = \frac{\tilde{g}^2}{M_Z^2} / \frac{g_2^2}{M_W^2} = \frac{\tilde{g}^2}{g_2^2} \times \frac{M_W^2}{M_Z^2} = \frac{1}{\cos^2 \theta} \times \cos^2 \theta = 1. \]  

(39)

Experimentally, \( \rho = 1 \) with high precision, and it is a strong evidence for the GWS theory.