1. In three spacetime dimensions (two space plus one time) an antisymmetric Lorentz tensor \( F^{\mu \nu} = -F^{\nu \mu} \) is equivalent to an axial Lorentz vector, \( F^{\mu \nu} = \epsilon^{\mu \nu \lambda} F_\lambda \). Consequently, in 3D one can make the photons massive without breaking the gauge invariance of the electromagnetic field \( A_\mu(x) \). Indeed, consider the following Lagrangian:

\[
\mathcal{L} = -\frac{1}{2} F_\lambda F^\lambda + \frac{m}{2} F_\lambda A^\lambda
\]  

where

\[
F_\lambda(x) = \frac{1}{2} \epsilon_{\lambda \mu \nu} F^{\mu \nu}(x) = \epsilon_{\lambda \mu \nu} \partial^\mu A^\nu(x),
\]

or in components, \( F_0 = -B, F_1 = +E_2, F_2 = -E_1 \).

(a) Show that the action \( S = \int d^3x \mathcal{L} \) is gauge invariant (although the Lagrangian (1) is not invariant).

(b) Write down the classical field equations — including both the Euler–Lagrange equations and the Bianchi identities — for the \( F_\lambda \) fields. Then show that these equations imply the Klein–Gordon equations \((\partial^2 + m^2) F_\lambda(x) = 0\).

Hint: in \( 2 + 1 \) dimension \( \epsilon^{\alpha \beta \gamma} \epsilon_\alpha^{\mu \nu} = g^{\beta \mu} g^{\gamma \nu} - g^{\beta \nu} g^{\gamma \mu} \).

(c) Write down the plane-wave solutions to the equation of motions and show that for each \( k^\mu = (+\omega_k, k) \) there is only one physical polarization that cannot be gauged away. Then argue — but without going through the gory details of quantizing the \( A^\mu(x) \) fields and expanding them into creation and annihilation operators — that the massive photons have only one 2D spin state, either only \( s = +1 \) or only \( s = -1 \), depending on the sign of \( m \).
(d) Write down the Noether stress-energy tensor for the theory in question, then add a suitable total divergence term of the form $\epsilon^{\mu\alpha\beta} \partial_\alpha K^\nu_\beta$ to make $T^{\mu\nu}$ gauge invariant and symmetric. The end result should be similar to the stress-energy tensor for the massless EM field,

$$T^{\mu\nu}_{\text{EM}} = -F^\mu{}_{\lambda} F^\nu_{\lambda} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} = +F^\mu F^\nu - \frac{1}{2} g^{\mu\nu} F_{\lambda} F^{\lambda} \quad \text{(in 3D).} \quad (3)$$

2. Next, consider the quantum theory of the massive EM fields in 3D. The Hamiltonian operator follows from the classical net energy $\int d^2x \, T^{0,0}$, thus in light of eq. (3)

$$\hat{H} = \int d^2x \left( \hat{T}^{0,0} = \frac{1}{2} (\hat{F}_1)^2 + \frac{1}{2} (\hat{F}_2)^2 + \frac{1}{2} (\hat{F}_0)^2 \right). \quad (4)$$

Note that this Hamiltonian is independent on the mass $m$. Instead, the mass affects the equal-time commutation relations of the quantum fields.

(a) Write down the canonical conjugate fields to the vector potentials $A_1(x)$ and $A_2(x)$, then use their relations to the $A_{\lambda}$ and $F_{\lambda}$ to show that in the quantum theory

$$\left[ \hat{F}_1(x, t), \hat{F}_2(y, \text{same } t) \right] = -im \delta^{(2)}(x - y). \quad (5)$$

(b) One of the classical equations of motion for the $F_{\mu}(x, t)$ fields is time-independent (i.e., does not involve time derivatives). Impose it as an operatorial identity expressing the quantum $\hat{F}_0(x, t)$ field in terms of space derivatives of the $\hat{F}_1(x, t)$ and $\hat{F}_2(x, t)$, then use it to derive the equal-time commutations relations between the $\hat{F}_0(x, t)$ and the $\hat{F}_{1,2}(y, t)$.

(c) Finally, use all these commutation relations and the Hamiltonian (4) to show that in the Heisenberg picture, the quantum $\hat{F}_{\lambda}(x)$ fields obey similar equations of motion to the classical fields.
3. Now let’s go back to $3+1$ dimensions and consider the superfluid liquid helium. Experimentally, the dispersion relation $\omega(k)$ for the quasiparticle excitations of the superfluid is obtained by scattering slow neutrons off the liquid helium. In most scattering events, one quasiparticle is produced, thus

$$|\text{neutron}(p); \text{BEC}\rangle \rightarrow |\text{neutron}(p'); \text{BEC} + 1\text{QP}(k')\rangle$$

$$\downarrow$$

$$k' = p - p' \quad \text{and} \quad \omega(k') = \frac{p^2}{2M_n} - \frac{p'^2}{2M_n}.$$  \hspace{1cm} \hspace{1cm} (6)

The interaction between a slow neutron and a helium atom is essentially pointlike and isotropic, so it may be described by a $\delta$-like potential. Thus, for a single neutron but a Dewar-full of liquid helium

$$\hat{H}_{\text{int}} = \sum_{\text{He atoms}} c \times \delta^{(3)}(\hat{X}_n - \hat{X}_{\text{atom}}).$$  \hspace{1cm} \hspace{1cm} (7)

Your task is to (a) rewrite this interaction Hamiltonian in terms of creation and annihilation operators for the Helium atoms, (b) show that the matrix element for the scattering process (6) is

$$\langle n(p'); \text{BEC} + 1\text{QP}(k') | \hat{H}_{\text{int}} | n(p); \text{BEC}\rangle =$$

$$= L^{-3/2} \delta_{p'+k',p} \times c \sqrt{n_s} \times \langle \text{BEC} + 1\text{QP}(k') | \hat{a}^\dagger_{k'} + \hat{a}_{-k'} | \text{BEC}\rangle$$

(where $n_s$ is the superfluid density), and (c) use the Bogolyubov transform to calculate the matrix element on the second line of eq. (8).

4. Finally, consider a non-abelian $SO(3)$ gauge theory coupled to a triplet of complex scalar fields $\Phi_a$,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + D_\mu \Phi_a^* D^\mu \Phi_a - V,$$  \hspace{1cm} \hspace{1cm} (9)

$$V = m^2 \Phi_a^* \Phi_a + \alpha (\Phi_a^* \Phi_a)^2 + \beta \sum_a |\epsilon_{abc} \Phi_b^* \Phi_c|^2,$$  \hspace{1cm} \hspace{1cm} (10)

where $a, b, c = 1, 2, 3$ and repeated indices are summed over.
(a) Identify all symmetries of this theory, global or local, discrete or continuous. For simplicity, skip the spacetime symmetries (Lorentz and translations) and focus on the internal symmetries only.

(b) Show that for $\alpha, \beta > 0$ but $m^2 < 0$, the scalar potential (10) has a continuous family of degenerate minima, and that all those minima are related by symmetries to

$$\langle \Phi \rangle = \frac{v}{\sqrt{2}} \times (0, 0, 1), \quad v = \sqrt{-\frac{m^2}{\alpha}}. \quad (11)$$

(c) Which symmetries are spontaneously broken by this vacuum expectation value (VEV)? Which symmetries remain unbroken? How many massless Goldstone bosons does this symmetry breaking calls for and what should be their quantum numbers? Which vector fields becomes massive by the Higgs mechanism and which remain massless?

(d) Expand the Lagrangian (9) in powers of $\Phi_a(x) - \langle \Phi_a \rangle$ and $A^a_\mu(x)$, write down the quadratic terms, and use them to determine the mass spectrum of the theory. Check that this spectrum agrees with predictions you have made in part (c).

Hint: split the complex fields $\Phi_a(x) - \langle \Phi_a \rangle$ into their real and imaginary parts.

(e) Generalize parts (a–c) to the $SO(N)$ gauge theory coupled to $N$ complex scalar fields.

Note: to generalize the third term in the potential (10), use

$$\sum_a |\epsilon_{abc}\Phi^*_b\Phi_c|^2 = (\Phi^*_a\Phi_a)^2 - (\Phi^*_a\Phi^*_a)(\Phi_b\Phi_b)$$

$$= 4(\Re\Phi_a\Re\Phi_a)(\Im\Phi_b\Im\Phi_b) - 4(\Re\Phi_a\Im\Phi_a)^2. \quad (12)$$