1. First, an exercise in bosonic commutation relations

\[ [\hat{a}_\alpha, \hat{a}_\beta] = 0, \quad [\hat{a}_\alpha, \hat{a}^\dagger_\beta] = 0, \quad [\hat{a}_\alpha, \hat{a}^\dagger_\beta] = \delta_{\alpha\beta}. \quad (1) \]

(a) Calculate the commutators
\[ [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta, \hat{a}_\gamma], \ [\hat{a}^\dagger_\alpha, \hat{a}_\beta, \hat{a}_\delta], \ [\hat{a}_\alpha, \hat{a}_\beta, \hat{a}^\dagger_\delta], \text{ and } [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta, \hat{a}^\dagger_\gamma, \hat{a}^\dagger_\delta, \hat{a}^\dagger_\mu, \hat{a}^\dagger_\nu]. \]

(b) For a single pair of \( \hat{a} \) and \( \hat{a}^\dagger \) operators, show that for any analytic function
\[ f(x) = f_0 + f_1 x + f_2 x^2 + \cdots, \]
\[ [\hat{a}, f(\hat{a}^\dagger)] = +f'(\hat{a}^\dagger) \quad \text{and} \quad [\hat{a}^\dagger, f(\hat{a})] = -f'(\hat{a}) \quad (2) \]

where \( f(\hat{a}) \overset{\text{def}}{=} f_0 + f_1 \hat{a} + f_2 (\hat{a})^2 + \cdots \) and likewise \( f(\hat{a}^\dagger) \overset{\text{def}}{=} f_0 + f_1 \hat{a}^\dagger + f_2 (\hat{a}^\dagger)^2 + \cdots. \)

(c) Show that
\[ e^{c\hat{a}^\dagger} e^{-c\hat{a}} = \hat{a}^\dagger + c, \quad e^{c\hat{a}^\dagger} \hat{a} e^{-c\hat{a}^\dagger} = \hat{a} - c, \quad \text{and hence for any analytic function } f, \]
\[ e^{c\hat{a}^\dagger} f(\hat{a}^\dagger) e^{-c\hat{a}^\dagger} = f(\hat{a}^\dagger + c) \quad \text{and} \quad e^{c\hat{a}^\dagger} f(\hat{a}) e^{-c\hat{a}^\dagger} = f(\hat{a} - c). \quad (3) \]

(d) Now generalize (b) and (c) to any set of creation and annihilation operators \( \hat{a}^\dagger_\alpha \) and \( \hat{a}_\alpha \). Show that for any analytic function \( f(\text{multiple } \hat{a}^\dagger_\alpha) \) of creation operators but not of the annihilation operators or a function \( f(\text{multiple } \hat{a}_\alpha) \) of the annihilation operators but not of the creation operators,
\[ [\hat{a}_\alpha, f(\hat{a}^\dagger)] = \frac{\partial f(\hat{a}^\dagger)}{\partial \hat{a}^\dagger_\alpha}, \quad [\hat{a}^\dagger_\alpha, f(\hat{a})] = -\frac{\partial f(\hat{a})}{\partial \hat{a}_\alpha}, \]
\[ \exp\left( \sum_\alpha c_\alpha \hat{a}_\alpha \right) f(\hat{a}^\dagger) \exp\left( -\sum_\alpha c_\alpha \hat{a}_\alpha \right) = f(\text{each } \hat{a}^\dagger_\alpha \rightarrow \hat{a}^\dagger_\alpha + c_\alpha), \quad (4) \]
\[ \exp\left( \sum_\alpha c_\alpha \hat{a}^\dagger_\alpha \right) f(\hat{a}) \exp\left( -\sum_\alpha c_\alpha \hat{a}^\dagger_\alpha \right) = f(\text{each } \hat{a}_\alpha \rightarrow \hat{a}_\alpha - c_\alpha). \]
2. An operator acting on identical bosons can be described in terms of \( N \)-particle wave functions (the \textit{first-quantized} formalism) or in terms of creation and annihilation operators in the Fock space (the \textit{second-quantized} formalism). This exercise is about converting the operators from one formalism to another.

The keys to this conversion are single-particle wave functions \( \phi_\alpha(x) \) of states \( |\alpha\rangle \) and \textit{symmetrized} \( N \)-particle states wave functions

\[
\phi_{\alpha\beta\ldots\omega}(x_1, x_2, \ldots, x_N) = \frac{1}{\sqrt{D}} \sum_{\text{all permutations of } (\alpha, \beta, \ldots, \omega)} \phi_\alpha(x_1) \times \phi_\beta(x_2) \times \cdots \times \phi_\omega(x_N)
\]

(5)

of \( N \)-boson states \( |\alpha, \beta, \ldots, \omega\rangle \). In eqs. (5), \( D \) is the number of \textit{distinct} permutations of single-particle states \( (\alpha, \beta, \ldots, \omega) \) and \( T \) is the number of trivial permutations. (A trivial permutation permutes states that happen to be the same, a distinct permutation permutes different states only.) In terms of the occupation numbers \( n_\gamma \),

\[
T = \prod_\gamma n_\gamma!, \quad D = \frac{N!}{T}.
\]

(6)

(a) Consider a generic \( N \)-particle quantum state \( |N; \psi\rangle \) with a totally symmetric wavefunction \( \Psi(x_1, \ldots, x_N) \). Show that the \((N + 1)\)-particle state \( |N + 1, \psi'\rangle = \hat{a}^\dagger_\alpha |N; \psi\rangle \) has wave function

\[
\psi'(x_1, \ldots, x_{N+1}) = \frac{1}{\sqrt{N+1}} \sum_{i=1}^{N+1} \phi_\alpha(x_i) \times \psi(x_1, \ldots, \hat{x}_i, \ldots, x_{N+1}).
\]

(7)

Hint: First prove this for wave-functions of the form (5). Then use the fact that states \( |\alpha_1, \ldots, \alpha_N\rangle \) form a complete basis of the \( N \)-boson Hilbert space.
(b) Show that the \((N - 1)\)-particle state \(|N - 1, \psi''\rangle = \hat{a}_\alpha |N; \psi\rangle\) has wave-function

\[
\psi''(x_1, \ldots, x_{N-1}) = \sqrt{N} \int d^3 x_N \phi_\alpha^*(x_N) \times \psi(x_1, \ldots, x_{N-1}, x_N).
\]  

(8)

Hint: the \(\hat{a}_\alpha\) is the hermitian conjugate of the \(\hat{a}_\alpha^\dagger\), so for any \(|N - 1, \tilde{\psi}\rangle\),

\[
\langle N - 1, \tilde{\psi} | \hat{a}_\alpha |N, \psi\rangle = \langle N, \psi | \hat{a}_\alpha^\dagger |N - 1, \tilde{\psi}\rangle^*.
\]

Next, consider one-body operators, \(i.e.\) additive operators acting on one particle at a time. In the first-quantized formalism they act on \(N\)-particle states according to

\[
\hat{A}_\text{net}^{(1)} = \sum_{i=1}^{N} \hat{A}_1(i^{th} \text{ particle})
\]  

(9)

where \(\hat{A}_1\) is some kind of a one-particle operator (such as momentum \(\hat{p}\), or kinetic energy \(\frac{1}{2m} \hat{p}^2\), or potential \(V(\hat{x})\), \textit{etc.}, \textit{etc.}). In the second-quantized formalism such operators become

\[
\hat{A}_\text{net}^{(2)} = \sum_{\alpha, \beta} \langle \alpha | \hat{A}_1 | \beta \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta.
\]  

(10)

(c) Verify that the two operators have the same matrix elements between any two \(N\)-boson states \(|N, \psi\rangle\) and \(|N, \tilde{\psi}\rangle\),

\[
\langle N, \tilde{\psi} | \hat{A}_\text{net}^{(1)} |N, \psi\rangle = \langle N, \tilde{\psi} | \hat{A}_\text{net}^{(2)} |N, \psi\rangle.
\]

Hint: use \(\hat{A}_1 = \sum_{\alpha, \beta} |\alpha\rangle \langle \alpha | \hat{A}_1 | \beta \rangle \langle \beta |\).

(d) Now let \(\hat{A}_\text{net}^{(2)}\), \(\hat{B}_\text{net}^{(2)}\), and \(\hat{C}_\text{net}^{(2)}\) be three second-quantized net one-body operators corresponding to the single-particle operators \(\hat{A}_1\), \(\hat{B}_1\), and \(\hat{C}_1\). Show that if \(\hat{C}_1 = [\hat{A}_1, \hat{B}_1]\) then

\[
\hat{C}_\text{net}^{(2)} = \left[\hat{A}_\text{net}^{(2)}, \hat{B}_\text{net}^{(2)}\right].
\]

Finally, consider two-body operators, \(i.e.\) additive operators acting on two particles at a time. Given a two-particle operator \(\hat{B}_2\) — such as \(V(\hat{x}_1 - \hat{x}_2)\) — the \textit{net} \(B\) operator acts in the first-quantized formalism according to

\[
\hat{B}_\text{net}^{(1)} = \frac{1}{2} \sum_{i \neq j} \hat{B}_2(i^{th} \text{ and } j^{th} \text{ particles}),
\]  

(11)

and in the second-quantized formalism according to

\[
\hat{B}_\text{net}^{(2)} = \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} (\langle \alpha | \otimes \langle \beta | \rangle \hat{B}_2(|\gamma\rangle \otimes |\delta\rangle)) \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta.
\]  

(12)
(e) Again, show these two operators have the same matrix elements between any two \(N\)-boson states, \(\langle N, \tilde{\psi} | \hat{A}_\text{net}^{(1)} | N, \psi \rangle = \langle N, \tilde{\psi} | \hat{A}_\text{net}^{(2)} | N, \psi \rangle\) for any \(\langle N, \tilde{\psi} |\) and \(|N, \psi\rangle\).

(f) Now let \(\hat{A}_1\) be a one-particle operator, let \(\hat{B}_2\) and \(\hat{C}_2\) be two-body operators, and let \(\hat{A}_\text{net}^{(2)}, \hat{B}_\text{net}^{(2)}, \) and \(\hat{C}_\text{net}^{(2)}\) be the corresponding second-quantized operators according to eqs. (10) and (12).

Show that if \(\hat{C}_2 = \left[ \left( \hat{A}_1(1^{\text{st}}) + \hat{A}_1(2^{\text{nd}}) \right), \hat{B}_2 \right]\) then \(\hat{C}_\text{net}^{(2)} = \left[ \hat{A}_\text{net}^{(2)}, \hat{B}_\text{net}^{(2)} \right].\)

3. The rest of this homework is about coherent states of harmonic oscillators and free quantum fields. Let us start with a harmonic oscillator with Hamiltonian \(\hat{H} = \hbar \omega \hat{a} \hat{a}^\dagger\).

(a) For any complex number \(\xi\) we define a coherent state \(|\xi\rangle \overset{\text{def}}{=} \exp(\xi \hat{a}^\dagger - \xi^* \hat{a}) |0\rangle\). Show that

\[
|\xi\rangle = e^{-|\xi|^2/2} e^{\xi \hat{a}^\dagger} |0\rangle \quad \text{and} \quad \hat{a} |\xi\rangle = \xi |\xi\rangle.
\] (13)

(b) Use \(\hat{a} |\xi\rangle = \xi |\xi\rangle\) to show that the (coordinate-space) wave function of a coherent state \(|\xi\rangle\) is a Gaussian wave packet of the same width as the ground state \(|0\rangle\). Also, show that the central position \(\bar{x}\) and the central momentum \(\bar{p}\) of this packet are related to the real and the imaginary parts of \(\xi\),

\[
\bar{x} = \sqrt{\frac{2 \hbar}{\omega m}} \times \text{Re} \xi, \quad \bar{p} = \sqrt{2 \hbar \omega m} \times \text{Im} \xi, \quad \xi = \frac{m \omega \bar{x} + i \bar{p}}{\sqrt{2m \omega \hbar}}. \tag{14}
\]

(c) Use \(\hat{a} |\xi\rangle = \xi |\xi\rangle\) and \(\langle \xi | \hat{a}^\dagger = \xi^* \langle \xi |\) to calculate the uncertainties \(\Delta x\) and \(\Delta p\) in a coherent state and verify their minimality: \(\Delta x \Delta p = \frac{\hbar}{2}\). Also, verify \(\delta n = \sqrt{n}\) where \(n \overset{\text{def}}{=} \langle \hat{n} \rangle = |\xi|^2\).

The coherent states are not stationary, they evolve with time but stay coherent — a coherent state \(|\xi_0\rangle\) at time \(t = 0\) becomes \(|\xi(t)\rangle\) at later times — while the central position \(\bar{x}\) and \(\bar{p}\) of the wave packet move according to the classical equations of motion for the harmonic oscillator.
(d) Check that such classical motion calls for \(\xi(t) = \xi_0 \times e^{-i\omega t}\), then check that the corresponding coherent state \(|\xi(t)\rangle\) obeys the time-dependent Schrödinger equation \(i\hbar \frac{d}{dt} |\xi(t)\rangle = \hat{H} |\xi(t)\rangle\).

(e) The coherent states are not quite orthogonal to each other. Calculate their probability overlaps \(|\langle \eta | \xi \rangle|^2\).

Now consider the coherent states of multi-oscillator systems such as quantum fields. In particular, let us focus on the creation and annihilation fields \(\hat{\Psi}^\dagger(x)\) and \(\hat{\Psi}(x)\) for non-relativistic spinless bosons.

(f) Generalize (a) and construct coherent states \(|\Phi\rangle\) which satisfy

\[
\hat{\Psi}(x) |\Phi\rangle = \Phi(x) |\Phi\rangle
\]

for any given classical complex field \(\Phi(x)\).

(g) Show that for any such coherent state, \(\Delta N = \sqrt{\tilde{N}}\) where

\[
\tilde{N} \overset{\text{def}}{=} \langle \Phi | \hat{N} | \Phi \rangle = \int dx |\Phi(x)|^2.
\]

(h) Let the Hamiltonian for the quantum non-relativistic fields be

\[
\hat{H} = \int dx \left( \frac{\hbar^2}{2M} \nabla \hat{\Psi}^\dagger(x) \cdot \nabla \hat{\Psi}(x) + V(x) \times \hat{\Psi}^\dagger(x) \hat{\Psi}(x) \right).
\]

Show that for any classical field configuration \(\Phi(x,t)\) obeying the classical field equation

\[
\frac{i\hbar}{\partial t} \Phi(x,t) = \left( -\frac{\hbar^2}{2M} \nabla^2 + V(x) \right) \Phi(x,t),
\]

the time-dependent coherent state \(|\Phi\rangle(t)\) satisfies the true Schrödinger equation

\[
i\hbar \frac{d}{dt} |\Phi\rangle = \hat{H} |\Phi\rangle.
\]

(i) Finally, show that the quantum overlap \(|\langle \Phi_1 | \Phi_2 \rangle|^2\) between two different coherent states is exponentially small for any macroscopic difference \(\delta \Phi(x) = \Phi_1(x) - \Phi_2(x)\) between the two field configurations.